Mini-Course Tensor Categories Wednesday

November 8, 2023

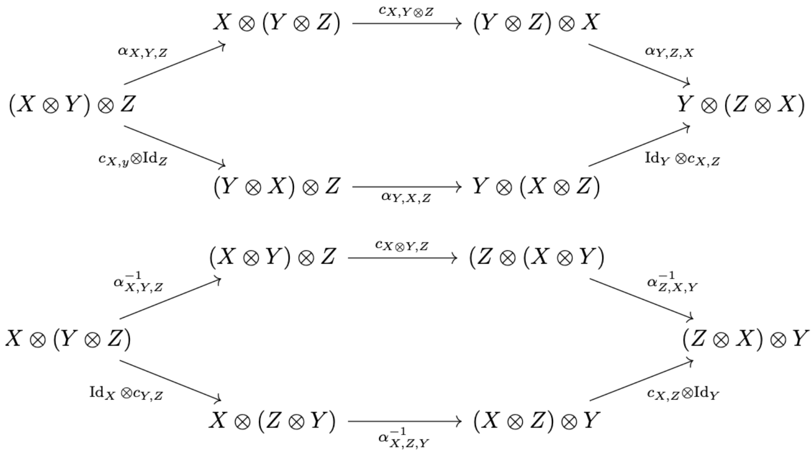
Review

Yesterday, we talked about

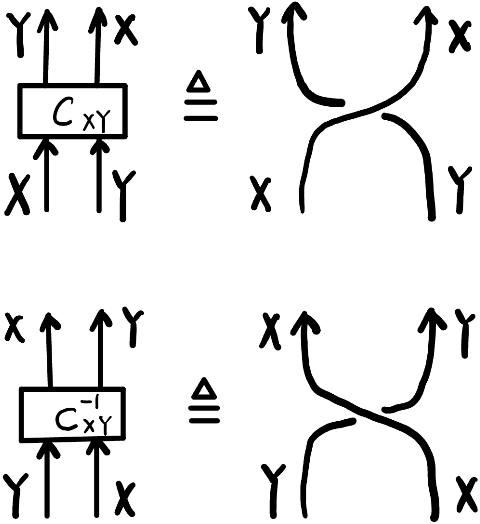
- Spherical categories
- Fusion categories, fusion rings
- Dimensions as ring characters

Braided Tensor Category

A braided tensor category is a tensor category $\mathcal{C}$ with a natural isomorphism $c_{X,Y} : X \otimes Y \simeq Y \otimes X$ called the braiding, such that the following two hexagon diagrams commute



Diagrams in Braided Tensor Categories



Symmetric Braided Fusion Categories

- Vec and Rep(G) are braided fusion categories. The braidings are the transpositions of two tensor factors.

  $\mathcal{C}$ is called symmetric if $c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}$, for all objects $X, Y \in \mathcal{C}$.

  Rep(G) is symmetric.

- Another braiding on Rep(G) : choose a central element such that $u^2 = 1$, and set

  $$\tilde{c}_{XY}(x \otimes y) = (-1)^{mn} y \otimes x$$

  if $ux = (-1)^m x, uy = (-1)^n y$. Denote this by Rep($G, u$).

- [Delinge] Symmetric fusion category $\mathcal{C}$ is equivalent to some Rep(G, u) as braided fusion category up to isomorphism.

Pointed Braided Fusion Categories

- Recall: Every pointed fusion category $\mathcal{C}$ is equivalent to a category Vec_G^ω.
- $\mathcal{C}$ is braided $\implies$ $G = A$ is abelian.
- Quadratic form q on G
  $q : G \to \mathbb{C}^\times$such that $q(g) = q\left(g^{-1}\right)$ and the symmetric function
  $$b(g,h) := \frac{q(gh)}{q(g)q(h)}$$
  is a bilinear.
- Pointed braided tensor category is uniquely determined by (A, q), where A is an abelian group and q is a quadratic form on A.

More Examples of Braided Fusion Categories

- [Siehler00] Tambara-Yamagami fusion category admits a braiding if and only if G is an elementary abelian 2-group.
- From twisted quantum double: $\mathrm{Rep}\,(D^{\omega} G)$, G finite, ω a 3-cocycle.
- From quantum group:

$$\mathfrak{g} \rightsquigarrow U_q\mathfrak{g} \overset{q=e^{\pi i/l}}{\rightsquigarrow} \mathrm{Rep}\,(U_q\mathfrak{g}) \overset{/\langle \mathrm{Ann}(Tr)\rangle}{\rightsquigarrow} \mathcal{C}(\mathfrak{g}, l)$$

See [EGNO Section 9.18.4] for a discussion on the construction of $\mathrm{Rep}\,(\mathcal{C}_q(G))$ from simply connected semisimple compact Lie group G.

Symmetric Center of a Braided Tensor Category

- The Müger center of a braided fusion category $\mathcal{C}$ is

$$\mathrm{Ob}(\mathcal{C}') = \{X \in \mathcal{C} : c_{X,Y} c_{Y,X} = \mathrm{Id}_{Y \otimes X}, Y \in \mathcal{C}\}$$

- The Müger center of $\mathrm{Rep}(G)$ is itself.
- Braided fusion category $\mathcal{C}$ is called nondegenerate if its Müger center is trivial.
- Spherical braided fusion category is called modular if it is nondegenerate.

The Drinfeld Center Construction

For simplicity, let's assume $\mathcal{C}$ is a strict tensor category.
An object of the $\mathcal{Z}(\mathcal{C})$ is a pair $(V, c_{-,V})$ where V is an object of $\mathcal{C}$ and $c_{-,V}$ is a family of natural isomorphisms

$$c_{X,V} : X \otimes V \to V \otimes X, \forall X \in \mathcal{C}$$

such that

$$c_{X \otimes Y,V} = (c_{X,V} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,V})$$

- A morphism from $(V, c_{-,V})$ to $(W, c_{-,W})$ is a morphism $f : V \to W$ in $\mathcal{C}$ such that for each object X of $\mathcal{C}$
$$(f \otimes \mathrm{id}_X)\, c_{X,V} = c_{X,W}\, (\mathrm{id}_X \otimes f)$$
- $\mathcal{Z}(\mathcal{C})$ is a tensor category with unit $(\mathbf{1}, \mathrm{id})$.
- $\mathcal{Z}(\mathcal{C})$ is a strict braided tensor category.

The Drinfeld Center Construction

- Let H be a f.d. Hopf algebra with invertible antipode.

  Let $\mathrm{D}(H) = (H^{\mathrm{op}})^* \bowtie H$ be the Drinfeld's Quantum Double of H.

- The braided tensor categories $\mathcal{Z}(\mathrm{Rep}(H))$ and $\mathrm{Rep}(\mathrm{D}(H))$ are equivalent.

- $\mathcal{Z}(\mathcal{C})$ of a fusion category $\mathcal{C}$ is a non-degenerate braided fusion category.

- $\mathcal{Z}(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$ is modular.

Let $\mathcal{C}$ be a braided tensor category, there is a braided tensor functor

$$G : \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \to \mathcal{Z}(\mathcal{C})$$

- A braided tensor category $\mathcal{C}$ is called factorizable if the above functor is an equivalence.

Equivalent Conditions for Non-degeneracy

Let $\mathcal{C}$ be a spherical braided fusion category.

[EGNO, Bruguières00, Kerler-Lyubashenko01]

The Following Are Equivalent:

- $\mathcal{C}$ is modular.
- $\mathcal{C}$ is factorizable.
- The S matrix of $\mathcal{C}$ is invertible.
- $\mathcal{C}$ has a special Hopf paring that is non-degenerate.

Also see [Shimizu19] for a general case.