Mini-Course on Tensor Categories Tuesday

November 7, 2023

Review

Yesterday, we discussed

- Definition of a tensor category and tensor functors.
- Rigid tensor category
  $X^* \otimes X \mapsto \mathbf{1} \mapsto X \otimes X^*$

Spherical Categories

- A pivotal monoidal category is a rigid monoidal category with a natural isomorphism $j : \mathrm{Id}_{\mathcal{C}} \to (-)^{**}$ of monoidal functors.
- Thus, $j_{V \otimes W} = j_V \otimes j_W$ for all $V, W \in \mathcal{C}$.
- Let $f : V \to V$ be a morphism in a pivotal category $\mathcal{C}$.

  The left and right quantum traces of f are

$$\mathrm{tr}^L(f) = \left(\mathbf{1} \xrightarrow{\mathrm{coev}} V \otimes V^* \xrightarrow{f \otimes \mathrm{id}} V \otimes V^* \xrightarrow{j_V \otimes \mathrm{id}} V^{**} \otimes V^* \xrightarrow{\mathrm{ev}_{V^*}} \mathbf{1}\right)$$

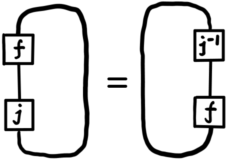
$$\mathrm{tr}^R(f) = \left(\mathbf{1} \xrightarrow{\mathrm{coev}} V^* \otimes V^{**} \xrightarrow{V^* \otimes j_V^{-1}} V^* \otimes V \xrightarrow{\mathrm{id} \otimes f} V^* \otimes V \xrightarrow{\mathrm{ev}} \mathbf{1}\right)$$

- If the left and right traces of every morphism are the same, then $\mathcal{C}$ is called a spherical category.

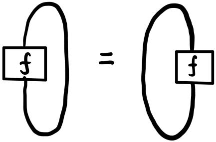
  In this case, denote traces and dimensions by $\mathrm{tr}(f)$ and $\dim(V)$.

Diagrams in Spherical Categories

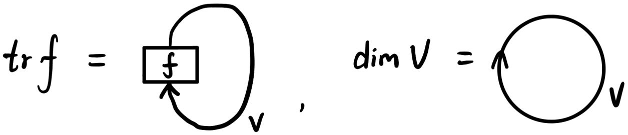
- If $\mathcal{C}$ is a strict tensor category, the spherical condition



- Pivotal strictification allows us to drop the j's:



Quantum Traces and Quantum Dimensions



- $\operatorname{tr}(f \otimes g) = \operatorname{tr} f \operatorname{tr} g, \quad \operatorname{tr}(f^*) = \operatorname{tr} f, \quad \operatorname{tr}(fg) = \operatorname{tr}(gf)$
- In particular,

  $\dim(V \otimes W) = \dim V \dim W, \quad \dim V^* = \dim V$
- Quantum Dimension of $\mathcal{C}$: $\dim(\mathcal{C}) = \sum_{X \in \operatorname{Irr}(\mathcal{C})} (\dim X)^2$

Additive Categories

We assume all tensor categories are additive, i.e.,

- Every set $\mathrm{Hom}(X, Y)$ is equipped with a structure of an abelian group such that composition of morphisms is biadditive with respect to this structure.
- There exists a zero object $0 \in \mathcal{C}$ such that $\mathrm{Hom}(0, 0) = 0$.
- (Existence of direct sums.)

  For any objects $X_1, X_2 \in \mathcal{C}$ there exists an object $Y \in \mathcal{C}$ and morphisms $p_1 : Y \to X_1, p_2 : Y \to X_2, i_1 : X_1 \to Y, i_2 : X_2 \to Y$ such that $p_1 i_1 = \mathrm{id}_{X_1}, p_2 i_2 = \mathrm{id}_{X_2}$, and $i_1 p_1 + i_2 p_2 = \mathrm{id}_Y$. The object Y is denoted by $X_1 \oplus X_2$, and is called the direct sum of X_1 and X_2.

We further assume $\mathcal{C}$ is abelian (one can talk about kernels and cokernels of morphisms).

Semisimplicity

- An object X in an tensor category $\mathcal{C}$ is simple if $\operatorname{End} X = \mathbb{C}$.
- An abelian category $\mathcal{C}$ is called semisimple if any object V is isomorphic to a direct sum of simple ones:

$$V \simeq \bigoplus_{i \in \operatorname{Irr} \mathcal{C}} N_i V_i$$

  where V_i are simple objects, $\operatorname{Irr} \mathcal{C}$ is the set of isomorphism classes of non-zero simple objects in $\mathcal{C}$, $N_i \in \mathbb{Z}_+$and only a finite number of N_i are non-zero.
- Example: $\operatorname{Rep}(G)$, G finite group, is semisimple by Maschke's Theorem:

  Every representation of a finite group G over $\mathbb{C}$ is a direct sum of irreducible representations.

Unitarity implies Semisimplicity

- A unitary tensor category is a rigid C^* tensor category with simple unit object such that all coherence isomorphisms are unitary.

- [Longo-Roberts 97] Let $\mathcal{C}$ be a unitary tensor category. Then for any object X in $\mathcal{C}$, $\mathsf{End}(X)$ is finite dimensional.

  In particular, $\mathcal{C}$ is semisimple.

  See [Yamagami 02] for a structural proof.

Deligne Tensor Product of Tensor Categories

Let $\mathcal{C}_1, \mathcal{C}_2$ be additive categories over $\mathbb{C}$.

Their Deligne tensor product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ is the category with

- Objects: finite sums of the form

$$\bigoplus X_i \boxtimes Y_i, \quad X_i \in \mathcal{C}_1, Y_i \in \mathcal{C}_2$$

- Morphisms:

$$\mathrm{Hom}_{\mathcal{C}_1 \boxtimes \mathcal{C}_2}\left(\bigoplus X_i \boxtimes Y_i, \bigoplus X_j' \boxtimes Y_j'\right) = \bigoplus_{i,j} \mathrm{Hom}\left(X_i, X_j'\right) \otimes \mathrm{Hom}\left(Y_i, Y_j'\right)$$

Fusion Category

A fusion category is a

- semisimple
- $\mathbb{C}$-linear
- rigid category with
- finite dimensional hom-sets, finitely many isomorphism classes of simple objects, and
- tensor unit $\mathbf{1}$ is simple.

Examples: $\mathrm{Rep}(G)$, Vec_G^{ω}.

- Ocneanu rigidity: [Etingof-Nikshych-Ostrik05] Up to equivalence, there is a finite number of fusion categories with a given fusion ring.
- **Question:** [Ostrik03] Are there finitely many equivalence classes of fusion categories with a given rank?

Fusion Ring

- Let $\mathcal{C}$ be a fusion category.
- Let $\mathrm{Irr}(\mathcal{C})$ denote the set of isomorphism classes of simple objects of $\mathcal{C}$.
- Rank of $\mathcal{C}$: $|\,\mathrm{Irr}(\mathcal{C})|$
- $\forall X, Y \in \mathrm{Irr}(\mathcal{C})$

$$X \otimes Y = \sum_{Z \in \mathrm{Irr}(\mathcal{C})} N_{X,Y}^{Z} Z$$

  called the fusion rules. N_{XY}^{Z} are called fusion coefficients.

- Let $\mathrm{Gr}(\mathcal{C})$ is the free abelian group with basis $\mathrm{Irr}(\mathcal{C})$.
- The map $* : \mathrm{Irr}(\mathcal{C}) \to \mathrm{Irr}(\mathcal{C})$ is an involution and $\mathbf{1}^* = \mathbf{1}$ and make $\mathrm{Gr}(\mathcal{C})$ into a fusion ring.

Fusion Ring

- Symmetries in the fusion coefficients:

  Note the fusion coefficients satisfy

  $$N_{XY}^{Z} = \dim \mathrm{Hom}(Z, X \otimes Y) = \dim \mathrm{Hom}\left(\mathbf{1}, X \otimes Y \otimes Z^*\right)$$
  $$N_{XY}^{Z} = N_{YX}^{Z} = N_{XZ^*}^{Y^*} = N_{X^*Y^*}^{Z^*}, \quad N_{XY}^{1} = \delta_{XY^*}.$$

- Quantum dimensions give a ring character:

  Recall $\dim(V) \in \mathrm{End}(\mathbf{1}) \cong \mathbb{C}$.

  - $\dim(V \otimes W) = \dim V \dim W$,
  - $\dim(V \oplus W) = \dim V + \dim W$
  - $\dim(\mathbf{1}) = 1$

- [Etingof-Gelaki-Nikshych-Ostrik] Dimensions of objects in a fusion category are algebraic integers in $\mathbb{C}$.

Frobenius-Perron Dimension

- Let N_X be the matrix with (Y, Z)-entry N_{XY}^Z for simple objects, which is called the fusion matrix.

  N_X is a squared matrix with nonnegative integers.

- Frobenius-Perron Theorem:

  The largest eigenvalue of any square matrix with positive entries is real, is of multiplicity one, and has an eigenvector with strictly positive entries.

- Let the Frobenius-Perron dimension of X be the maximal eigenvalue of the fusion matrix N_X, $X \in \mathrm{Irr}(\mathcal{C})$.

- The Frobenius-Perron dimension of $\mathcal{C}$ is $\mathrm{FPdim}\,\mathcal{C} = \sum_{X \in \mathrm{Irr}(\mathcal{C})} (\mathrm{FPdim}\,X)^2$.

Frobenius-Perron Dimension

- Frobenius-Perron dimensions give a character of the fusion ring:

  $\mathrm{FPdim}(V \otimes W) = \mathrm{FPdim}(V) \cdot \mathrm{FPdim}(W)$,
  $\mathrm{FPdim}(V \oplus W) = \mathrm{FPdim}(V) + \mathrm{FPdim}(W)$, $\mathrm{FPdim}(\mathbf{1}) = 1$

- [Etingof-Nikshych-Ostrik05] Let $\mathcal{C}$ be a fusion category.

  $\forall X \in \mathrm{Irr}(\mathcal{C})$, $|X|^2 \leq \mathrm{FPdim}(X)^2$. Thus $\dim(\mathcal{C}) \leq \mathrm{FPdim}(\mathcal{C})$.

- If $\mathcal{C}$ is unitary, $\dim(X) > 0$ for all X.

- Moreover, $\dim(X) = \mathrm{FPdim}(X)$ for all X.

- A fusion category $\mathcal{C}$ is called pseudo-unitary if $\dim(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C})$.

- [Etingof-Nikshych-Ostrik05] A pseudo-unitary fusion category admits a unique spherical structure s.t., $\dim_j(X) = \mathrm{FPdim}(X)$ for every simple object X.

Some Classification Results by FP-dimensions

Let $\mathcal{C}$ be a fusion category.

- [Etingof-Nikshych-Ostrik05] FPdim X is a cyclotomic integer ≥ 1.
  If $\mathrm{FPdim}(X) < 2$, for some $X \in \mathrm{Irr}(\mathcal{C})$, then $\mathrm{FPdim}(X) = 2\cos(\pi/n)$, for some integer $n \geq 3$.

- [Calegari-Morrison-Snyder11] . Let X be an object in a fusion category such that $\mathrm{FPdim}\, X$ belongs to the interval $(2, 76/33]$. Then $\mathrm{FPdim}\, X$ is equal to one of:

$$\frac{\sqrt{7}+\sqrt{3}}{2}, \quad \sqrt{5}, \quad 1+2\cos\left(\frac{2\pi}{7}\right), \quad \frac{1+\sqrt{5}}{\sqrt{2}}, \quad \frac{1+\sqrt{13}}{2}.$$

- $\mathcal{C}$ is called integral if FPdim $X \in \mathbb{Z}$, $\forall X \in \mathcal{C}$.
  $\mathcal{C}$ is integral iff it is equivalent to the category of f.d representations of a f.d semisimple quasi-Hopf algebra over $\mathbb{C}$.
- [Gelaki-Nikshych08] Every fusion category of odd integer Frobenius-Perron dimension is integral.

Fibonacci Category

Fibonacci Category

- Simple Object: $\mathbf{1}$, τ
- Rank: 2
- Fusion Rules: $\tau \otimes \tau = \mathbf{1} \oplus \tau$
- Quantum Dimensions: $d_{\mathbf{1}} = 1, \quad d_{\tau} = \frac{1+\sqrt{5}}{2}$

Ising Category

Ising Category

- Simple Object: $\mathbf{1}$, σ, ψ
- Rank: 3
- Fusion Rules:

$$\sigma^2 = 1 + \psi, \quad \psi^2 = 1, \quad \psi\sigma = \sigma\psi = \sigma$$

- Quantum Dimensions: $d_{\mathbf{1}} = \mathbf{1}, \quad d_\psi = \mathbf{1}, \quad d_\sigma = \sqrt{2}$.

Observation: The objects $\mathbf{1}$ and ψ generate a fusion subcategory. Recall these are invertible objects in the category.

Pointed Fusion Categories

- Recall an object X in $\mathcal{C}$ is invertible if

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$$
$$\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$$

  are isomorphisms.

  Notice X is invertible iff $\mathrm{FPdim}(X) = 1$.

- A fusion category $\mathcal{C}$ is pointed if every simple object of $\mathcal{C}$ is invertible. A pointed subcategory of $\mathcal{C}$ is denoted by $\mathcal{C}_{\mathrm{pt}}$.

- Recall Vec_G^ω is pointed.

- Every pointed fusion category is equivalent to a category Vec_G^ω, for some 3-cocycle ω, where G is the group of invertible objects of $\mathcal{C}$.

Examples of Fusion Categories

- Given a finite group G, the Tambara-Yamagami Fusion Category has label set $G \cup \{m\}$, where G is a group and $m \notin G$, with fusion rules

  $$g \otimes h = gh, \quad m \otimes g = g \otimes m = m, \quad m^2 = \bigoplus_{g \in G} g$$

  for $g, h \in G$.

  When $G = \mathbb{Z}_2$, this is the Ising fusion category.

- $\mathrm{Rep}(S_3)$

  Simple objects: $1, \chi, V$

  Fusion rules: $\chi V = V\chi = V, \chi^2 = 1, V^2 = 1 + \chi + V$.

Observation: In both categories, all but one simple object is invertible.

Near-group Fusion Categories

A near-group category is a fusion category $\mathcal{C}$ in which all but one simple object is invertible.

- Simple objects: $G \cup \{\rho\}$
- Fusion rules: $g\rho = \rho g = \rho$ for all $g \in G$,
- $\rho \otimes \rho = n'\rho + \sum_{g \in G} g$.

$$d_\rho = \frac{n' + \sqrt{n'^2 + 4n}}{2}$$

- Denote near-group category with the above fusion rules by $G + n'$.
- [Evans-Gannon '14] Given a near-group category of type $G + n'$, the only possible values for n' are $0, n-1$, or $n' \in n\mathbb{Z}$, where $n = |G|$.
- [Tambara-Yamagami '98] The fusion categories of type $G + 0$ are completely classified.

Examples of Non-unitary Spherical Fusion Category

Yang-Lee Theory

- Simple Object: 1, $\bar{\tau}$
- Rank: 2
- Fusion Rules: $\bar{\tau} \otimes \bar{\tau} = 1 \oplus \bar{\tau}$
- Quantum Dimensions: $d_1 = 1, \quad d_\tau = \frac{1-\sqrt{5}}{2}$

$\mathrm{Vec}_{\mathbb{Z}_2}^{-}$: Category of $\mathbb{Z}_2$-graded vector spaces

- Simple Object: 1, a^-
- Rank: 2
- Fusion Rules: $a^- \otimes a^- = \mathbf{1}$
- Quantum Dimensions: $d_1 = 1, \quad d_{a^-} = -1$

Remark: There is another fusion category $\mathrm{Vec}_{\mathbb{Z}_2}$ with simple object a and a spherical j s.t. $d_a = 1$.