Mini-Course Tensor Categories Thursday

November 9, 2023

Review

Yesterday, we talked about

- Braided Tensor categories
- Symmetric centers of braided fusion categories
- Drinfeld Centers $\mathcal{Z}(\mathcal{C})$
- Example: $\mathcal{Z}\left(\mathrm{Rep}\left(\mathbb{Z}_2\right)\right)$ gives us Toric Code

  $\mathrm{FPdim}(\mathcal{Z}(\mathcal{C})) = (\mathrm{FPdim}(\mathcal{C}))^2$

Frobenius Algebras in Tensor Categories

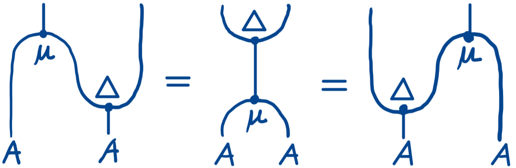
Let $\mathcal{C}$ be a (strict) tensor category.

A Frobenius algebra in $\mathcal{C}$: $(A, \mu, \eta, \Delta, \varepsilon)$, where

$A \in \mathcal{C}$, $\mu : A \otimes A \to A$, $\eta : \mathbf{1} \to A$, $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbf{1}$,

- (A, μ, η) an associative unital algebra,
- (A, Δ, ε) is a coassociative counital coalgebra,such that the Frobenius condition holds:

$$(1 \otimes \mu) \circ (\Delta \otimes 1) = \Delta \circ \mu = (\mu \otimes 1) \circ (1 \otimes \Delta)$$



Module Categories

Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}, \alpha, l, r)$ be a monoidal category. A left module category over $\mathcal{C}$ is a category $\mathcal{M}$ equipped with

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$
- a module associativity constraint

  $$m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M), \quad X, Y \in \mathcal{C}, M \in \mathcal{M}$$

- unit constraint

  $$l_M : \mathbf{1} \otimes M \overset{\sim}{\to} M$$

  such that the pentagon diagram and the triangle diagram commute.

Module Categories

- the pentagon diagram

$$
\begin{array}{ccccc}
 & & (X \otimes Y) \otimes (Z \otimes M) & & \\
 & \overset{m_{X \otimes Y, Z, M}}{\nearrow} & & \overset{m_{X, Y, Z \otimes M}}{\searrow} & \\
((X \otimes Y) \otimes Z) \otimes M & & & & X \otimes (Y \otimes (Z \otimes M)) \\
\alpha_{X,Y,M} \otimes \mathrm{id}_M \downarrow & & & & \uparrow \mathrm{id}_X \otimes m_{Y,Z,M} \\
(X \otimes (Y \otimes Z)) \otimes M & & \xrightarrow[m_{X, Y \otimes Z, M}]{} & & X \otimes ((Y \otimes Z) \otimes M)
\end{array}
$$

- Triangle Axiom

$$
\begin{array}{ccc}
(X \otimes \mathbf{1}) \otimes M & \xrightarrow{m_{X,\mathbf{1},M}} & X \otimes (\mathbf{1} \otimes M) \\
 r_X \otimes \mathrm{id}_M \searrow & & \swarrow \mathrm{id}_X \otimes l_M \\
 & X \otimes M &
\end{array}
$$

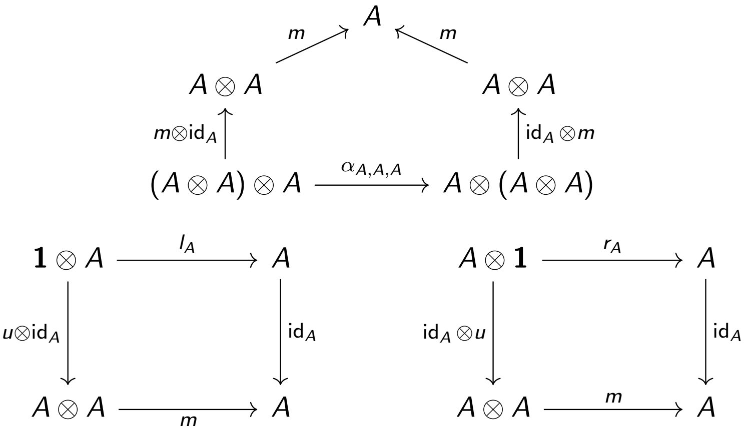
- We can further define $\mathcal{C}$-module functor
$$\zeta_{X,M} : F(X \otimes M) \to X \otimes F(M), \quad X \in \mathcal{C}, M \in \mathcal{M}$$
satisfying certain conditions.

Algebra in a Tensor Category

An algebra in $A \in \mathcal{C}$ is a triple (A, m, u), where

- multiplication morphism $m : A \otimes A \to A$
- unit morphism $u : \mathbf{1} \to A$

such that the following diagrams commute



Examples of Algebras in Monoidal Categories

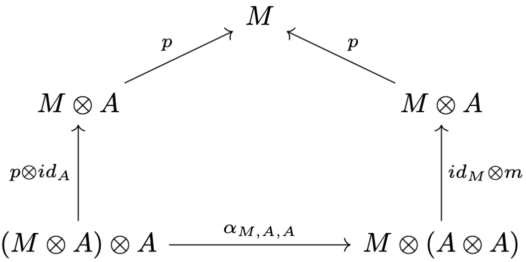
Recall our convention: Tensor Category = Monoidal + $\mathbb{C}$-linear

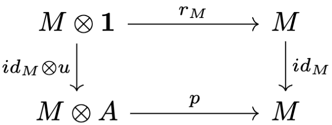
Monoidal Category $(\mathcal{C}, \otimes, \mathbf{1})$	Algebra Objects
$(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$	Rings
$\mathsf{Vec}_{\mathrm{fd}}, \mathsf{Vec}$	(f.d.) unital associative algebra
Vec_G	G-graded algebra
Rep(G)	Fun(G) - algebra of functions on G
$(\mathsf{End}_{\mathcal{C}}, \circ, \mathrm{Id})$	Monads

Modules over Algebras

A right A-module in $\mathcal{C}$ is a pair (M, p), where

- $M \in \mathcal{C}$
- $p : M \otimes A \to M$ is a morphism such that the following diagrams commute





Category of A-modules

- Let $\mathrm{Mod}_{\mathcal{C}}(A)$ be the category of right A-modules in $\mathcal{C}$.
- Then $\mathrm{Mod}_{\mathcal{C}}(A)$ is a left $\mathcal{C}$-module category with the action:

$$(X \otimes M) \otimes A \xrightarrow{\alpha_{X,M,A}} X \otimes (M \otimes A) \xrightarrow{\mathrm{id}_X \otimes p} X \otimes M$$

- Two algebras A and B in $\mathcal{C}$ are Morita equivalent if

$$\mathrm{Mod}_{\mathcal{C}}(A) \cong \mathrm{Mod}_{\mathcal{C}}(B)$$

  are equivalent $\mathcal{C}$-module categories.
- Let $\mathcal{M}$ be a semisimple indecomposable module category over $\mathcal{C}$.
- [Ostrik03] There exist semisimple indecomposable algebra $A \in \mathcal{C}$ such that $\mathcal{M} \cong \mathrm{Mod}_{\mathcal{C}}(A)$ as module categories.

Categorical Morita Equivalence

Reference:[Müger03], [Etingof-Nikshych-Ostrik05]

Let $\mathcal{C}$ be a fusion category.

- Let $\mathcal{M}$ be an indecomposable right $\mathcal{C}$-module category $\mathcal{M}$.

  The category of $\mathcal{C}$-module endofunctors $\mathcal{C}^*_{\mathcal{M}}$ on $\mathcal{M}$ is a fusion category, which is called the dual of $\mathcal{C}$ with respect $\mathcal{M}$.

- $\mathcal{C}$ and $\mathcal{D}$ are categorically Morita equivalent if

  $$\mathcal{C}^*_{\mathcal{M}} \cong \mathcal{D}$$

  for some indecomposable right $\mathcal{C}$-module category $\mathcal{M}$.
- Example:

  $$(\mathrm{Vec}_G)^*_{\mathrm{Vec}} \cong \mathrm{Rep}(G),$$

  thus Vec_G and $\mathrm{Rep}(G)$ are Morita equivalent.

- If $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent, then

  $$\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{D})$$

Categorical Morita Equivalence

Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories.

- [Etingof-Nikshych-Ostrik]
  $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if and only if

$$\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{D})$$

  as braided fusion categories.

- A fusion category $\mathcal{C}$ is called group-theoretical if it is Morita equivalent to a pointed fusion category.

Further Reference on Frobenius Algebras in Tensor Categories

- [Müger03] From subfactors to categories and topology I: Frobenius algebras in and Morita equivalence of tensor categories
- [Bischoff-Kawahigashi-Longo-Rehren15] Tensor Categories and Endomorphisms of von Neumann Algebras
- [Carqueville-Runkel-Schaumann18] Orbifolds of Reshetikhin–Turaev TQFTs
- [Mulevičius-Runkel23] Constructing modular categories from orbifold data