Mini-Course on Tensor Categories Monday

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Landscape: vN vs C*

Goal: Transfer(ence) of subfactor techniques to C*-algebras

von Neumann Algebras

Classification:

▶ [MvN43, Con76] Injective factors:

$$M \otimes \mathcal{R} \cong M$$

∃! hyperfinite II_1-factor:

$$\mathcal{R} \cong \mathcal{R} \otimes \mathcal{R}$$

Subfactors: $N \subset M$ (Survey: [Pop23])

▶ [Jon83] Index Rigidity Theorem:

$[M : N] \in \{4\cos^2 \frac{\pi}{n}\}_{n \geq 3} \cup [4, \infty]$.

$N \subset M$ ⇝ **quantum symmetry**:

$\{N \subset M\} \leftrightarrow \{\mathcal{C} \curvearrowright N\text{+generator}\}$

C*-algebras

Elliott Program: (Survey: [Whi23])

▶ 'Many hands' [GLN14, EGLN15, TWW17]

Classify simple amenable C*-algs by K-theory and traces

Feat: $\mathcal{Z}, \mathcal{O}_2, \mathcal{O}_n, \mathcal{O}_\infty$, AF, A_θ, ...

C*-inclusions: $A \subset B$ [HPN23]

♡ C*-algs have QuSymmetry!

♣ Characterization of framework

♢ Classification of QuDynamics?

♠ Interactions with K-theory

Natural habitat for Unitary Tensor Categories (UTC)

Discrete & compact groups:

- ∘ $\mathsf{Hilb}_\mathsf{f}(\Gamma)$
- • $\mathsf{Hilb}_\mathsf{f}(\Gamma, \omega)$, $[\omega] \in H^3(G, \mathbb{T})$
- • • $\mathsf{Rep}_\mathsf{f}(G)$

$\bigcap$

Discrete/compact quantum gps [NT13]**:**

- • $\left\{\mathbb{G}\right\} \overset{\text{T-K-W}}{\longleftrightarrow} \left\{ \underbrace{\mathsf{Rep}_\mathsf{f}(\mathbb{G}) \overset{\otimes}{\to} \mathsf{Hilb}_\mathsf{f}}_{\text{Fiber functor}} \right\}$

$\bigcap$

Subfactors: The standard invariant

[Ocn88, Pop95a, Jon22, Müg03]

$$\left\{N \subset M\right\} \leftrightarrow \left\{ \underbrace{\mathcal{C}_{N\subset M} \overset{\otimes}{\to} \mathsf{Bim}(N)}_{\text{generalized fiber functor}} + \underbrace{Q := {}_N L^2(M)_N}_{\text{Q-system [Lon84]}} \right\}$$

- ▷ $(N \subset M) \cong (N \subset N \rtimes Q)$
- ▶ Every $\mathcal{C}$ comes from $L\mathbb{F}_\infty \cong N \subset M \cong L\mathbb{F}_\infty$ [PS03]

Subfactors and their standard invariants

Realization/Crossed Products: [ILP98, JP19]

$$\{N \subset M\} \underset{\rtimes Q}{\longleftarrow} \left\{\mathcal{C}_{N\subset M} \overset{\otimes}{\to} \mathrm{Bim}(N) \ + \ Q := {}_N L^2(M)_N\right\}$$

$$\rightsquigarrow (N \subset M) \cong (N \subset N \rtimes Q)$$

Subfactor Classification

P1 Analytic/Dynamical: Construct & classify $\{\mathcal{C} \curvearrowright N\}$,

P2 Algebraic: Construct & classify Q-sys/W*-algebras $\{Q \in \mathcal{C}\}$.

Example (Hyperfinite Subfactors $\mathcal{R} \cong N \subset M \cong \mathcal{R}$ [Pop94, Pop95b])

Standard invariant is complete for amenable hyperfinite subfactors!

C* Quantum dynamics

C*-algebras have quantum symmetry too!

- Every UTC from some C*-inclusion [HHP20]
- C*-algebras are *Q-system complete* [CHPJP22]

C*-inclusions

Standard invariants transfer to C*-inclusions [HPN23]:

$$\Big\{\underbrace{\mathcal{C} \overset{\otimes}{\to} \mathsf{Bim}(A) \;+\; \mathcal{C}\text{-graded } \mathrm{C}^*\text{-Alg } \mathbb{B}}_{\text{C* Quantum Dynamics}}\Big\} \quad \rightsquigarrow \quad \Big\{A \overset{E}{\subset} A \rtimes_{\mathsf{r}} \mathbb{B}\Big\}$$

Known: Largest class $\{A \subset B\}$ determined by standard invariant

Future problems

- Construct/classify C* Qudynamics (on classifiables)
- Robustness of classifiable C* by discrete extensions

Content of Minicourse

1. Introduction and UTC examples
2. Fundamentals of tensor categories
3. Unitarity in tensor categories, Index theory and Concrete examples
4. Q-systems & algebra objects, and the standard invariant
5. Actions of UTCs on C*-algebras and their crossed products by example

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Tensor Category

Example: Rep(G), category of finite dimensional representations of a finite group G over $\mathbb{C}$.

- Objects: representations of G over $\mathbb{C}$
- Morphisms: Interwiners

A tensor category $\mathcal{C}$ over $\mathbb{C}$ is category that is

- monoidal: $(\otimes, \mathbf{1})$
- $\mathbb{C}$-linear: $\mathrm{Hom}(X, Y)$ is $\mathbb{C}$-vector space and composition is bilinear.

Tensor Categories

A tensor category $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, l, r)$ is a $\mathbb{C}$-linear category $\mathcal{C}$ with

- a tensor product functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

- a unit object

$$\mathbf{1} \in \mathcal{C}$$

- an associativity constraint α,

$$\alpha : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$$

- a left unit constraint

$$l_X : \mathbf{1} \otimes X \xrightarrow{\sim} X,$$

- a right unit constraint

$$r_X : X \otimes \mathbf{1} \xrightarrow{\sim} X,$$

Tensor Categories

and the data $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, l, r)$ satisfies

- the Pentagon Axiom

$$
\begin{array}{ccccc}
 & & (W \otimes X) \otimes (Y \otimes Z) & & \\
 & \nearrow^{\alpha_{W\otimes X, Y, Z}} & & \searrow^{\alpha_{W, X, Y\otimes Z}} & \\
((W \otimes X) \otimes Y) \otimes Z & & & & W \otimes (X \otimes (Y \otimes Z)) \\
\alpha_{W,X,Y} \otimes \mathrm{id}_Z \downarrow & & & & \uparrow \mathrm{id}_W \otimes \alpha_{X,Y,Z} \\
(W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow{\alpha_{W, X\otimes Y, Z}} & & W \otimes ((X \otimes Y) \otimes Z)
\end{array}
$$

- Triangle Axiom

$$
\begin{array}{ccc}
(X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\alpha_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\
 & {}_{r_X \otimes \mathrm{id}_Y}\searrow \quad \swarrow_{\mathrm{id}_X \otimes l_Y} & \\
 & X \otimes Y &
\end{array}
$$

- The tensor category is said to be strict if α, l, r are all identities.

Example: Category of Vector Spaces

- $\mathcal{C} = \mathsf{Vec}_{\mathbb{C}}$, the category of vector spaces over $\mathbb{C}$.
- $\otimes$ is the tensor product of vector spaces over $\mathbb{C}$.
- $\mathbf{1} = \mathbb{C}$ is the ground field $\mathbb{C}$.
- associativity

  $$\alpha((u \otimes v) \otimes w) = u \otimes (v \otimes w),$$

  for $u \in U, v \in V, w \in W$.
- unit constraints

  $$l(1 \otimes v) = v = r(v \otimes 1)$$

Vec_G^ω: Category of G-graded Vector Spaces

- Objects: G-graded f.d. vector spaces $V = \bigoplus_{g\in G} V_g$.
- Morphisms: linear maps which preserve the grading.
- If $V = \oplus_{g\in G} V_g$ and $W = \oplus_{g\in G} W_g$, then

$$(V \otimes W)_g = \bigoplus_{h\in G} V_h \otimes W_{h^{-1}g}$$

- $\mathbf{1} = \mathbb{C}_e$, and associativity

$$\alpha_{V,W,Z} : (V \otimes W) \otimes Z \to V \otimes (W \otimes Z)$$
$$(v_g \otimes w_h) \otimes z_k \mapsto \omega(g,h,k) v_g \otimes (w_h \otimes z_k)\,,$$

$g, h, k \in G, v_g \in V_g, w_h \in W_h, z_k \in Z_k$.

- ω is a 3-cocycle: $\omega : G \times G \times G \to \mathbb{C}^\times$ such that

$$\omega(ab,c,d)\omega(a,b,cd) = \omega(a,b,c)\omega(a,bc,d)\omega(b,c,d)$$

Tensor Categories from Representations

$\mathrm{Rep}(G)$, G a group.

- Objects: representations of G over $\mathbb{k}$,
- Morphisms: interwiners,
- $\otimes$ is the tensor product of representations

$$\rho_{V\otimes W}(g) := \rho_V(g) \otimes \rho_W(g)$$

$\mathrm{Rep}(\mathfrak{g})$, $\mathfrak{g}$ a Lie algebra over $\mathbb{C}$

- $\otimes$ is defined by

$$\rho_{V\otimes W}(a) = \rho_V(a) \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \rho_W(a)$$

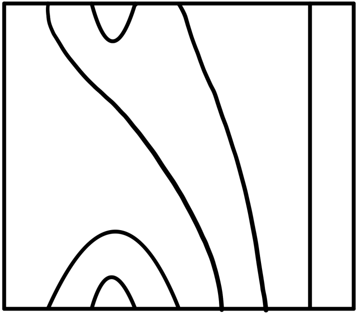
Temperley-Lieb Diagrams

Let t be an indeterminant over $\mathbb{C}$ and $d = \left(t + t^{-1}\right)$.

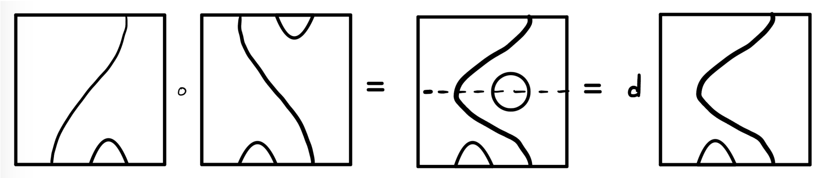
Let $m, n \in \{0, 1, 2, \ldots\}$ and $m - n$ even.

The (m, n)-TL diagrams

Figure: A $(5, 7)$-TL Diagram



Composing Temperley-Lieb Diagrams



Temperley-Lieb Categories

The generic Temperley-Lieb category is a tensor category with

- objects: elements of $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.
- morphisms

  If $n - m$ is odd, then $\mathrm{Hom}(m, n)$ is the 0 -vector space.

  If $n - m$ even, $\mathrm{Hom}(m, n)$ is the $\mathbb{C}(t)$-vector space with basis the set of equivalence classes of (m, n)-TL diagrams.

- tensor product

  on objects: $n \otimes n' = n + n'$.

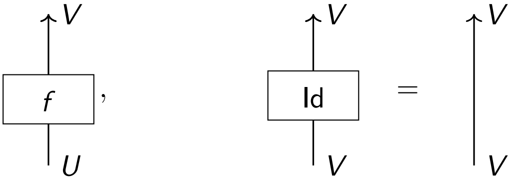
  on morphisms: horizontal juxtaposition.

  Tensoring an (n, m)-TL diagram with an (n', m')-diagram gives a $(n + n', m + m')$- TL diagram.

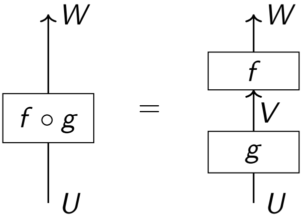
Graphical Calculus for Morphisms

Let $\mathcal{C}$ be a strict tensor category.

- $f : U \to V$

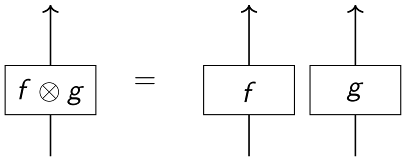


- $f \circ g$

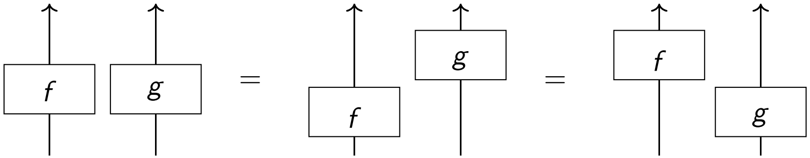


Graphical Calculus for Morphisms

- $f \otimes g$



- $f \otimes g = (f \circ \mathrm{id}) \otimes (\mathrm{id} \circ g) = (\mathrm{id} \circ f) \otimes (g \circ \mathrm{id})$



Dagger Category

We say that $\mathcal{C}$ is a dagger category if and only if for each $X, Y \in \mathcal{C}$ there is an anti-linear map

$$\dagger : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(Y, X)$$

such that

- $\left(f^{\dagger}\right)^{\dagger} = f$, for any $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}(X, Y)$.
- $(f \circ g)^{\dagger} = g^{\dagger} \circ f^{\dagger}$
- $(f \otimes g)^{\dagger} = f^{\dagger} \otimes g^{\dagger}$

We say that $f \in \mathrm{Hom}(X, Y)$ is unitary if it is invertible with $f^{-1} = f^{\dagger}$.

- Examples:

  Hilb, the category of complex Hilbert spaces.

  Hilb(Γ, ω), the category of ω-twisted complex Γ- graded Hilbert spaces.

C^* Category

A dagger category $\mathcal{C}$ is called a C^*-category if

- For every $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}(X, Y)$, there is a $g \in \mathrm{End}(X)$ such that $f^\dagger \circ f = g^\dagger \circ g$.
- The function $\|\cdot\| : \mathrm{Hom}(X, Y) \to [0, \infty]$ defined by

  $$\|f\|^2 = \sup\left\{ |\lambda| \geq 0 \mid f^\dagger \circ f - \lambda \mathrm{id}_X \text{ is not invertible } \right\}$$

  is a complete norm on $\mathrm{Hom}(X, Y)$.
- $\|f \circ g\| \leq \|f\| \cdot \|g\|$
- $\left\|f^\dagger \circ f\right\| = \|f\|^2$

Tensor Functor

Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}, \alpha, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes', \mathbf{1}', \alpha', l', r')$ be tensor categories.
A (strong) tensor functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ with a isomorphism $F_0 : \mathbf{1}' \to F(\mathbf{1})$ in $\mathcal{D}$ and with a natural isomorphism

$$F_2(X, Y) : F(X) \otimes' F(Y) \to F(X \otimes Y)$$

such that

$$\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{\alpha'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
\downarrow {\scriptstyle F_2(X,Y) \otimes' \mathrm{id}_{F(Z)}} & & \downarrow {\scriptstyle \mathrm{id}_{F(X)} \otimes' F_2(Y,Z)} \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\
\downarrow {\scriptstyle F_2(X \otimes Y, Z)} & & \downarrow {\scriptstyle F_2(X, Y \otimes Z)} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha_{X,Y,Z})} & F(X \otimes (Y \otimes Z))
\end{array}$$

Tensor Functor

$$\begin{array}{ccc} \mathbf{1}' \otimes' F(X) & \xrightarrow{l'_{F(X)}} & F(X) \\ {\scriptstyle F_0 \otimes \mathrm{id}_{F(X)}} \downarrow & & \uparrow {\scriptstyle F(l_X)} \\ F(\mathbf{1}) \otimes' F(X) & \xrightarrow[F_2(\mathbf{1}, X)]{} & F(\mathbf{1} \otimes X), \end{array} \qquad \begin{array}{ccc} F(X) \otimes' \mathbf{1}' & \xrightarrow{r'_{F(X)}} & F(X) \\ {\scriptstyle \mathrm{id}_{F(X)} \otimes F_0} \downarrow & & \uparrow {\scriptstyle F(r_X)} \\ F(X) \otimes' F(\mathbf{1}) & \xrightarrow[F_2(X, \mathbf{1})]{} & F(X \otimes \mathbf{1}) \end{array}$$

commute.

A natural transformation of tensor functors
$\eta : (F, F_0, F_2) \to (G, G_0, G_2)$ is a natural transformation
$\eta : F \to G$ such that $\eta_{\mathbf{1}}$ is an isomorphism, and

$$\eta_{X \otimes Y} F_2(X, Y) = G_2(X, Y) \left(\eta_X \otimes' \eta_Y \right)$$

Examples of Tensor Functors

Let A be a unital C^*-algebra and Γ be a discrete group acting on A.

Then we have a tensor functor F

$$F : \mathsf{Hilb}(\Gamma) \to \mathsf{Bim}(A)$$
$$\mathbb{C}_g \mapsto {}_gA$$

where ${}_gA$ is a is A as a right Hilbert A-module and

$$a \triangleright b = g^{-1}(a)b$$

The tensorator for F is

$$F_2^{g,h} : F(g) \otimes_A F(h) \cong F(gh)$$
$$a \otimes b \mapsto h^{-1}(a)b.$$

This can be generalized to anomalous actions on a C^*-algebra.

Duality in $\mathrm{Vec}_{\mathbb{C}}$

Consider $\mathrm{Vec}_{\mathbb{C}}$, the category of f.d. vector spaces over $\mathbb{C}$.

- $\forall V \in \mathrm{Ob}(\mathrm{Vec}_{\mathbb{C}})$, $\exists V^*$, and morphisms

$$\begin{aligned} \mathrm{ev}_V &: V^* \otimes V \to \mathbb{C}, \\ \mathrm{coev}_V &: \mathbb{C} \to V \otimes V^*, \end{aligned}$$

- ev_V is the evaluation map,
- $\mathrm{coev}_V(1) := \sum v_i \otimes v^i$, $\{v_i\}$ and $\left\{v^i\right\}$ are dual bases in V and V^*.

Duality in Tensor Categories

- Let $\mathcal{C}$ be a tensor category and $X \in \mathcal{C}$. A left dual of X is an object X^* with

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}, \quad \mathrm{coev}_X : \mathbf{1} \to X \otimes X^*,$$

  such that the composition

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*$$

  are identities.
- Similarly, one can define right dual $({}^*X, \mathrm{ev}'_X, \mathrm{coev}'_X)$ of X.
- A tensor category $\mathcal{C}$ is called rigid if every object of $\mathcal{C}$ has right and left duals.

Dual Morphism

If $X, Y \in \mathcal{C}$ which have left duals X^*, Y^* and $f : X \to Y$ is a morphism.

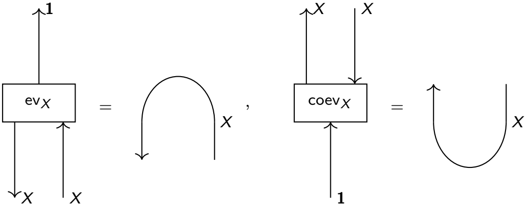
Define the left dual $f^* : Y^* \to X^*$ of f by

$$f^* := Y^* \xrightarrow{\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X} Y^* \otimes X \otimes X^* \xrightarrow{\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}} Y^* \otimes Y \otimes X^* \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}} X^*.$$

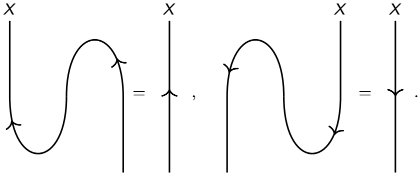
Similarly, one can define the right dual ${}^*f : {}^*Y \to {}^*X$ of f.

Graphical Calculus for Rigidity

- ev_X and coev_X



- The graphical form of rigidity axioms



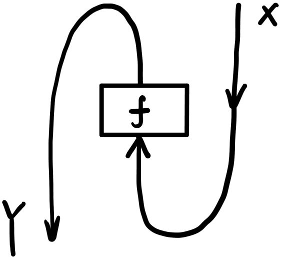
Recall

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*$$

Graphical Calculus for Rigidity

- The left dual $f^* : Y^* \to X^*$



Recall

$$f^* := Y^* \xrightarrow{\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X} Y^* \otimes X \otimes X^* \xrightarrow{\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}} Y^* \otimes Y \otimes X^* \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}} X^*.$$

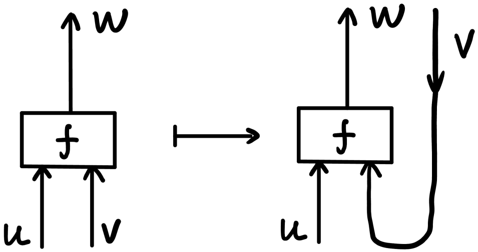
- Let $f : V \to W, g : U \to V$ be morphisms in $\mathcal{C}$, then

$$(f \circ g)^* = g^* \circ f^*$$
$$(\mathrm{id}_V)^* = \mathrm{id}_{V^*}$$

Useful Adjunctions

For any family U, V, W of objects of $\mathcal{C}$, we have natural bijections

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}\left(U, W \otimes V^*\right)$$



and

$$\mathrm{Hom}\left(U^* \otimes V, W\right) \cong \mathrm{Hom}(V, U \otimes W)$$

Examples of Rigid Tensor Categories

- Vec_G^ω with normalized cocycle ω.

  $\mathbb{C}_g^* = {}^*\mathbb{C}_g = \mathbb{C}_{g^{-1}}$

  Normalize the coevaluation morphisms of $\mathbb{C}_g$ to be the identities. Then

  $$\mathrm{ev}_{\mathbb{C}_g} = \omega\left(g, g^{-1}, g\right) \mathrm{id}_{\mathbf{1}}$$

- Rep_G. For a finite dimensional representation V,
  - the dual representation V^* is the usual dual space
  - G-action is given $\rho_{V^*}(g) = \left(\rho_V(g)^{-1}\right)^*$

Invertible Objects

- Let $\mathcal{C}$ be a rigid tensor category.
- An object X in $\mathcal{C}$ is invertible if

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$$
$$\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$$

  are isomorphisms.
- Examples:
  - The objects $\mathbb{C}_g$ in Vec_G^ω are invertible.
  - The invertible objects in $\mathrm{Rep}(G)$ are the 1-dimensional representations of G.
- The invertible objects of $\mathcal{C}$ form a tensor subcategory $\mathrm{Inv}(\mathcal{C})$ of $\mathcal{C}$.

Plan for Tomorrow

- Pivotal structure, Categorical trace, Spherical categories
- Fusion categories
- ...