Super-modular categories from near-group centers

Qing Zhang

Purdue University

Joint work with Eric Rowell and Hannah Solomon

Motivation

Modular category with modular data (S, T).

- (S, T) gives a projective representation $\bar{\rho}$ of $\mathrm{SL}_2(\mathbb{Z})$.
- [Ng-Schauenburg '10] If $N = \mathrm{ord}(T)$, $\bar{\rho}$ factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.
- [Ng-Rowell-Wang-Wen '22] Reconstruction of modular data from irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

Super-modular category with $(\hat{S}, \hat{T})$.

- $(\hat{S}, \hat{T}^2)$ gives a projective representation $\hat{\rho}$ of $\Gamma_\theta < \mathrm{SL}_2(\mathbb{Z})$.
- [Bonderson-Rowell-Wang-Z'19] $\hat{\rho}\,(\Gamma_\theta)$ is a finite group assuming minimal modular extension.
- [Cho-Kim-Seo-You '22] Unrealized (S, T) for super-modular categories.

Modular Tensor Categories

- A pre-modular category $\mathcal{C}$ over $\mathbb{C}$ is

  monoidal: $(\otimes, \mathbf{1})$,

  semisimple: $X \cong \oplus_i m_i X_i$,

  linear: $\mathrm{Hom}(X,Y) \in \mathrm{Vec}_{\mathbb{C}}$,

  rigid: $X^* \otimes X \mapsto \mathbf{1} \mapsto X \otimes X^*$,

  finite rank: finitely many simple objects X_i; $\mathbf{1}$ is simple,

  braided: $\beta_{X,Y} : X \otimes Y \cong Y \otimes X$,

  spherical: $j : \mathrm{id}_{\mathcal{C}} \xrightarrow{\cong} (-)^{**}$.

  Plus compatibility conditions.

Modular Tensor Categories

- A premodular category $\mathcal{C}$ over $\mathbb{C}$ is a spherical braided fusion category.
  - $T_{X,Y} := \delta_{X,Y}\theta_X, X, Y \in \mathrm{Irr}(\mathcal{C})$.

    Ribbon structure $\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$.
    If $X \in \mathrm{Irr}(\mathcal{C})$, then θ_X is a nonzero scalar multiple of id_X.
  - $S_{X,Y} := \mathrm{tr}\left(\beta_{Y,X^*} \circ \beta_{X^*,Y}\right), X, Y \in \mathrm{Irr}(\mathcal{C})$.
- $\mathcal{C}$ is modular if S is invertible.
- The pair of matrices (S, T) is called the modular data of $\mathcal{C}$.
- Dimensions: $d_X := S_{\mathbf{1},X}$, $\dim(\mathcal{C}) = D^2 = \sum_X d_X^2$.
- Fusion rule: $X \otimes Y = \oplus N_{XY}^Z Z$.

  Fusion coefficients: $N_{X,Y}^Z = \dim \mathrm{Hom}(X \otimes Y, Z)$.

Examples of Modular Tensor Categories

- Pointed: $\mathcal{C}(A, Q)$, abelian group A, non-deg. quadratic form Q on A.

- From quantum groups:

$$\mathfrak{g} \rightsquigarrow U_q\mathfrak{g} \overset{q=e^{\pi i/l}}{\rightsquigarrow} \mathrm{Rep}\,(U_q\mathfrak{g}) \overset{/\langle \mathrm{Ann}(\mathrm{Tr}) \rangle}{\rightsquigarrow} \mathcal{C}(\mathfrak{g}, l)$$

- The Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$.

  Objects of $\mathcal{Z}(\mathcal{C})$ are (Z, γ), where Z is an object of $\mathcal{C}$ and γ is half braiding.

$\mathrm{SL}_2(\mathbb{Z})$ Representation

Given a modular category with (unnormalized) modular data (S, T).

- $S^4 = \dim(\mathcal{C})^2\,\mathrm{Id}$, $(ST)^3 = p^+\,S^2$, where $p^\pm = \sum_i \theta_i^\pm d_i^2$.
- $\mathrm{SL}_2(\mathbb{Z}) = \langle \mathfrak{s}, \mathfrak{t} \rangle$, where $\mathfrak{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathfrak{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- $\mathfrak{s}^4 = \mathrm{Id}$, $(\mathfrak{s}\mathfrak{t})^3 = \mathfrak{s}^2$.
- $\mathfrak{s} \mapsto S$, $\mathfrak{t} \mapsto T$ gives a projective representation $\bar{\rho}_{\mathcal{C}}$ of $\mathrm{SL}_2(\mathbb{Z})$.
- $H < \mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if H contains some $\Gamma(n) := \{A \in \mathrm{SL}(2, \mathbb{Z}) : A \equiv I \pmod{n}\}$.
- $\mathrm{SL}_2(\mathbb{Z})/\Gamma(n) \cong \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for all $n > 1$.
- [Ng-Schauenburg '10]

  If $N = \mathrm{ord}(T)$,

  - $\bar{\rho}_{\mathcal{C}}$ factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.
  - $\mathbb{Q}(S) \subset \mathbb{Q}(\zeta_N)$, $N = \mathrm{ord}(T)$.

$\mathrm{SL}_2(\mathbb{Z})$ Representation (continued)

- $s := \dfrac{1}{\sqrt{\dim(\mathcal{C})}}\, S$ and $t := \dfrac{1}{\gamma}\, T$, where γ is any third root of the multiplicative central charge $\xi = p^+(\mathcal{C})/\sqrt{\dim(\mathcal{C})}$.
- $\mathfrak{s} \mapsto s$, $\mathfrak{t} \mapsto t$ gives a linear representation ρ of $\mathrm{SL}_2(\mathbb{Z})$.
- [Dong-Lin-Ng '15]

  If $n = \mathrm{ord}(t)$,

  - ρ factors through $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$
  - $\mathrm{im}(\rho) \subset \mathsf{GL}_r(\mathbb{Q}(\zeta_n))$

Modular Data from $\mathrm{SL}_2(\mathbb{Z})$ Representations

Can we classify modular data from finite-dimensional reps of $\mathrm{SL}_2(\mathbb{Z})$?

- ρ factors through $\mathsf{SL}_2(\mathbb{Z}/n\mathbb{Z})$.
- Chinese Reminder Theorem $\rightarrow$ $\mathsf{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$.
- [Nobs 1976, Nobs and Wolfart 1976] Irreducible representations of $\mathsf{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$ are classified using subrepresentations of Weil representations.
- [Ng-Wang-Wilson '23] Every finite-dimensional congruence representation of $\mathrm{SL}_2(\mathbb{Z})$ is symmetrizable.
- [Ng-Rowell-Wang-Wen '22] Reconstruction of modular data from irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

  Classification up to modular data, rank $= 6$.

Super-modular categories

Example (sVec)

- Two simple objects: $\mathbf{1}$ and fermion f.
- $S_{\mathrm{sVec}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\qquad T_{\mathrm{sVec}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $f \otimes f = \mathbf{1}$, $c_{f,f} = -1$.
- sVec is symmetric since $c_{X,Y} c_{Y,X} = \mathrm{Id}_{Y \otimes X}$ for all X, Y.

- For $\mathcal{D} \subset \mathcal{B}$ BFCs, the Müger centralizer of $\mathcal{D}$ in $\mathcal{B}$ is

$$C_{\mathcal{B}}(\mathcal{D}) = \{X \in \mathcal{B} : c_{X,Y} c_{Y,X} = \mathrm{Id}_{Y \otimes X}, Y \in \mathcal{D}\}$$

- The Müger center $\mathcal{B}' = C_{\mathcal{B}}(\mathcal{B})$.
- If $\mathcal{B}$ is modular, then $\mathcal{B}' \simeq$ Vec.

Definition (Super-modular Category)

Unitary pre-modular category $\mathcal{B}$ is super-modular if $\mathcal{B}' \cong \mathrm{sVec}$.

Super-modular Categories from $U_q(\mathfrak{sl}_2)$

Example

$\mathrm{PSU}(2)_{4m+2}$: subcategory of $\mathrm{SU}(2)_{4m+2}$ is super-modular.

- $2m+2$ simple objects: $\{Y_0 = \mathbf{1}, Y_1 = X_2, \ldots, Y_{2m+1} = X_{4m+2}\}$.
- $S_{ij} = \dfrac{\sin(\frac{(2i+1)(2j+1)\pi}{(4m+4)})}{\sin(\frac{\pi}{(4m+4)})}$ and $\theta_j = e^{\frac{\pi i(j^2+j)}{2m+2}}$.
- $Y_1 \otimes Y_k \cong Y_{k-1} \oplus Y_{k+1} \oplus Y_k$ for $k \ll \infty$.
- $\mathrm{PSU}(2)'_{4m+2} = \langle Y_{2m+1} \rangle \cong \mathrm{sVec}$.

Where do Super-modular Categories Come from?

- split super-modular: $\mathcal{C} \boxtimes \mathrm{sVec}$, where $\mathcal{C}$ is modular.

- Let $(\mathcal{C}, f)$ be a spin modular category, where $\mathcal{C}$ is modular and $f \in \mathcal{C}$ is a fermion.
  - Then $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$, where $\mathcal{C}_0 = C_{\mathcal{C}}(\langle f \rangle)$ is super-modular.
  - Also, $\dim(\mathcal{C}_0) = \dim(\mathcal{C})/2$.

- [Johnson-Freyd-Reutter '23] Every super-modular category $\mathcal{B}$ can be embedded in a modular category $\mathcal{C}$ of dimension $\dim(\mathcal{C}) = 2\dim(\mathcal{B})$.

Projective Representation of Γ_θ

- $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \hat{S}$, $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{T}$, where $\hat{S}$ is symmetric and nondegenerate.
- $\mathrm{SL}_2(\mathbb{Z}) = \langle \mathfrak{s}, \mathfrak{t} \rangle$. The assignment $\mathfrak{s} \to \hat{S}$, $\mathfrak{t}^2 \to \hat{T}^2$ gives a projective representation $\hat{\rho}$ of $\Gamma_\theta = \langle \mathfrak{s}, \mathfrak{t}^2 \rangle \subset \mathrm{SL}_2(\mathbb{Z})$.

Theorem (Bonderson-Rowell-Wang-Z'19)

For a super-modular category, $\hat{\rho}(\Gamma_\theta)$ is a finite group, with kernel a congruence subgroup.

Super-modular Categories by Rank

- Rank ≤ 6.
  [BGNPRW '20] $\mathrm{PSU}(2)_{4m+2}$ for $m = 0, 1, 2$ are the only prime super-modular categories up to fusion rules.

- Rank $= 8$.
  [Bruillard-Plavnik-Rowell-Z '21] With bounded fusion coefficients and dimensions, the prime super-modular categories are $\mathrm{PSU}(2)_{14}$, $[\mathrm{PSU}(2)_6 \otimes \mathrm{PSU}(2)_6]_{\mathbb{Z}_2}$, $[\mathrm{SO}(12)_2]_0$.

- Rank $= 10$.
  [Cho-Kim-Seo-You '22] Reconstrucion of $(\hat{S}, \hat{T}^2)$ from congruence representations of Γ_θ, adapted from classifying modular categories.

  Two families of unrealized modular data for super-modular categories.

New (S,T) for Super-modular Categories

[Cho-Kim-Seo-You '22]

Unrealized data for rank 10 super-modular categories

$$\hat{S} = \begin{bmatrix} 1 & 4+\sqrt{15} & 5+\sqrt{15} & 3+\sqrt{15} & 3+\sqrt{15} \\ 4+\sqrt{15} & 1 & 5+\sqrt{15} & -3-\sqrt{15} & -3-\sqrt{15} \\ 5+\sqrt{15} & 5+\sqrt{15} & -5-\sqrt{15} & 0 & 0 \\ 3+\sqrt{15} & -3-\sqrt{15} & 0 & \frac{1}{2}(1+\sqrt{5})(3+\sqrt{15}) & -\frac{2\sqrt{30}(4+\sqrt{15})}{5+\sqrt{5}} \\ 3+\sqrt{15} & -3-\sqrt{15} & 0 & -\frac{2\sqrt{30(4+\sqrt{15})}}{5+\sqrt{5}} & \frac{1}{2}(1+\sqrt{5})(3+\sqrt{15}) \end{bmatrix}$$

$$\hat{T}^2 = \mathrm{Diag}\left[1, 1, e^{2i\pi/3}, e^{4i\pi/5}, e^{-4i\pi/5}\right]$$

$$\hat{S} = \begin{bmatrix} 1 & 5+2\sqrt{6} & 3+\sqrt{6} & 3+\sqrt{6} & 4+2\sqrt{6} \\ 5+2\sqrt{6} & 1 & 3+\sqrt{6} & 3+\sqrt{6} & -4-2\sqrt{6} \\ 3+\sqrt{6} & 3+\sqrt{6} & -3-\sqrt{6}-i\sqrt{6\left(2\sqrt{6}+5\right)} & -3-\sqrt{6}+i\sqrt{6\left(2\sqrt{6}+5\right)} & 0 \\ 3+\sqrt{6} & 3+\sqrt{6} & -3-\sqrt{6}+i\sqrt{6\left(2\sqrt{6}+5\right)} & -3-\sqrt{6}-i\sqrt{6\left(2\sqrt{6}+5\right)} & 0 \\ 4+2\sqrt{6} & -4-2\sqrt{6} & 0 & 0 & 4+2\sqrt{6} \end{bmatrix}$$

$$\hat{T}^2 = \mathrm{Diag}\left[1, 1, -1, -1, e^{2i\pi/3}\right]$$

Near-group Categories

- A near-group category is a fusion category $\mathcal{C}$ in which all but one simple object is invertible.
- Simple objects: $G \cup \{\rho\}$
- Fusion rules:
  - $g\rho = \rho g = \rho$ for all $g \in G$,
  - $\rho \otimes \rho = n'\rho + \sum_{g \in G} g$.
- $d_\rho = \dfrac{n' + \sqrt{n'^2 + 4n}}{2}$

Denote near-group category with the above fusion rules by $G + n'$.

- [Evans-Gannon '14]

  Given a near-group category of type $G + n'$, the only possible values for n' are $0, n-1$, or $n' \in n\mathbb{Z}$, where $n = |G|$.
- [Tambara-Yamagami '98] The fusion categories of type $G + 0$ are completely classified.

Near-group Categories of type $G + n$

- [Izumi '01, Evans-Gannon '14] Near-group categories of type $G + n$ are determined by $(G, a, b, c, \langle\, ,\, \rangle)$, where $c \in \mathbb{T}$, $a : G \to \mathbb{T}$, $b : G \to \mathbb{C}$ and $\langle\, ,\, \rangle$ is a non-degenerate symmetric bicharcter satisfying

$$a(0) = 1, \quad a(x) = a(-x), \quad a(x+y)\langle x, y\rangle = a(x)a(y), \quad \sum_{a \in G} a(x) = \sqrt{n}c^{-3},$$

$$b(0) = -\frac{1}{d}, \quad \sum_y \overline{\langle x, y\rangle} b(y) = \sqrt{n}c\overline{b(x)}, \quad a(x)b(-x) = \overline{b(x)},$$

$$\sum_x b(x+y)\overline{b(x)} = \delta_{y,1} - \frac{1}{d}, \quad \sum_x b(x+y)b(x+z)\overline{b(x)} = \overline{\langle y, z\rangle} b(y)b(z) - \frac{c}{d\sqrt{n}}.$$

  Solutions exist for $G = \mathbb{Z}_n, 1 \leqslant n \leqslant 13$.
  $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_6$.

  No solutions for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

- [Schopieray '22] If G is an an elementary abelian 2-group of order n, then the near-group fusion ring $R(G + n)$ is categorifiable if and only if $G = \mathbb{Z}_2$ or $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.

- [Budinski '21] Solutions exist for $G = \mathbb{Z}_n, 1 \leqslant n < 29$.

Centers of Near-group Categories of Type $G+n$

[Izumi '01] Formula for modular data using the tube algebra construction.

- To obtain (S,T), we need to solve the equations for (ξ,τ,ω), where $\xi : G \to \mathbb{T}$, $\tau \in G$, and $\omega \in \mathbb{T}$ such that

$$\sum_g \xi(g) = \sqrt{n}\omega^2 a(\tau)c^3 - nd^{-1}, \quad \xi(\tau - g) = \omega c^4 a(g)a(\tau - g)\overline{\xi(g)}$$

$$\bar{c}\sum_k b(g+k)\xi(k) = \omega^2 c^3 a(\tau)\overline{\xi(g+\tau)} - \sqrt{n}d^{-1}$$

$$\sum_k \xi(k)b(k-g)b(k-h) = c^{-2}b(g+h-\tau)\xi(g)\xi(h)\overline{a(g-h)} - c^2 d^{-1}$$

- Then T and S matrices are given by

$$T = \mathrm{diag}\left[\langle g,g\rangle, \langle h,h\rangle, \langle k,l\rangle, \omega_j\right]$$

$$S = \frac{1}{\lambda}\begin{bmatrix} \left\langle g,g'\right\rangle^{-2} & (d+1)\left\langle g,h'\right\rangle^{-2} & (d+2)\overline{\langle g,k'+l'\rangle} & d\left\langle g,\tau_{j'}\right\rangle \\ (d+1)\left\langle h,g'\right\rangle^{-2} & \left\langle h,h'\right\rangle^{-2} & (d+2)\overline{\langle h,k'+l'\rangle} & -d\left\langle h,\tau_{j'}\right\rangle \\ (d+2)\overline{\langle k+l,g'\rangle} & (d+2)\overline{\langle k+l,h'\rangle} & S_{(k,l),(k',l')} & \mathbf{0} \\ d\left\langle \tau_j,g'\right\rangle & -d\left\langle \tau_j,h'\right\rangle & \mathbf{0} & S_{j,j'} \end{bmatrix}$$

with 4 classes of simple objects:
(1) $\mathfrak{a}_g, g \in G$, (2) $\mathfrak{b}_h, h \in G$, (3) $\mathfrak{c}_{l,k} = \mathfrak{c}_{k,l}, l,k \in G, l \neq k$,
(4) $\mathfrak{d}_j$, where j corresponds to a triple (ξ_j,τ_j,ω_j).

Centers of Near-group Categories of Type $G+n$

- [Evans-Gannon '14] conjectured the modular data can be obtained by metric groups (G, q_1) and (Γ, q_2), where
  - $|\Gamma| = |G| + 4$,
  - $\frac{1}{\sqrt{|G|}} \sum_{g \in G} q_1(g) = -\frac{1}{\sqrt{|\Gamma|}} \sum_{\gamma \in \Gamma} q_2(\gamma)$, if $|G|$ is odd.

  Then the lower right corner for (S, T) can be replaced by

  $$t_{(\tau,\gamma),(\tau,\gamma)} = \langle \tau, \tau \rangle \langle \gamma, \gamma \rangle', \quad (\tau \in G, \gamma \in \Gamma \backslash \{0\})$$
  $$s_{(\tau,\gamma),(\tau',\gamma')} = -d\, \overline{\langle \tau, \tau' \rangle} \left(\langle \gamma, \gamma' \rangle' + \overline{\langle \gamma, \gamma' \rangle'} \right), \quad (\tau, \tau' \in G, \gamma, \gamma' \in \Gamma \backslash \{0\})$$

  - Verified for $n \leq 13$.
- Generalized and further verified in [Grossman-Izumi '20] and [Budinski '21].
- Central charge $\neq 1$ modular data from the conjecture [Bonderson-Rowell-Wang] Realization by zesting and gauging.

Modular Data when $G = \mathbb{Z}/6\mathbb{Z}$

- [Evans-Gannon '14] Up to equivalence, there are 4 near-group categories when $G = \mathbb{Z}/6\mathbb{Z}$.

	m	c	$b(1), b(2), b(3)$
J_6^1	1	ζ_{24}^{1}	$2.95526, \; 0.0553542, -0.785398$
$\overline{J_6^1}$	-1	ζ_{24}^{-1}	$-2.95526, -0.0553542, \; 0.785398$
J_6^2	5	ζ_{24}^{5}	$2.91503, -1.59091, \; 2.35619$
$\overline{J_6^2}$	-5	ζ_{24}^{-5}	$-2.91503, \; 1.59091, -2.35619$

$\sqrt{n}b(x) = \exp(ij(x)),\ j(x) \in [-\pi, \pi].$

- There are 27 solutions to the triples (ω, ξ, τ)

#	ω	τ	ξ
1	6	0	$0.837758, -2.16701, 2.73289, 2.40855, 1.03703, 2.79533$
2	24	0	$-1.36136, 0.0503995, -0.101868, -2.93215, -0.526451, 2.46287$
3	36	0	$2.40855, 1.03703, 2.79533, 0.837758, -2.16701, 2.73289$
4	54	0	$-2.93215, -0.526451, 2.46287, -1.36136, 0.0503995, -0.101868$
5	55	1	$-1.60604, 1.60604, -1.0472, -1.60604, 1.60604, -1.0472$
6	31	1	$2.42435, 1.34557, 0.428593, -1.39623, -1.11705, 1.24692$
7	31	1	$-1.39623, -1.11705, 1.24692, 2.42435, 1.34557, 0.428593$
8	19	1	$-2.51206, -1.25786, -2.13757, -0.499749, 3.01302, 2.55645$
9	19	1	$-0.499749, 3.01302, 2.55645, -2.51206, -1.25786, -2.13757$
10	4	2	$-0.101868, -2.93215, -0.526451, 2.46287, -1.36136, 0.0503995$
11	16	2	$2.79533, 0.837758, -2.16701, 2.73289, 2.40855, 1.03703$
12	34	2	$2.46287, -1.36136, 0.0503995, -0.101868, -2.93215, -0.526451$
13	46	2	$2.73289, 2.40855, 1.03703, 2.79533, 0.837758, -2.16701$
14	15	3	$-1.0472, -1.60604, 1.60604, -1.0472, -1.60604, 1.60604$
15	39	3	$2.55645, -2.51206, -1.25786, -2.13757, -0.499749, 3.01302$
16	39	3	$-2.13757, -0.499749, 3.01302, 2.55645, -2.51206, -1.25786$
17	51	3	$1.24692, 2.42435, 1.34557, 0.428593, -1.39623, -1.11705$
18	51	3	$0.428593, -1.39623, -1.11705, 1.24692, 2.42435, 1.34557$
19	4	4	$-0.526451, 2.46287, -1.36136, 0.0503995, -0.101868, -2.93215$
20	16	4	$-2.16701, 2.73289, 2.40855, 1.03703, 2.79533, 0.837758$
21	34	4	$0.0503995, -0.101868, -2.93215, -0.526451, 2.46287, -1.36136$
22	46	4	$1.03703, 2.79533, 0.837758, -2.16701, 2.73289, 2.40855$
23	55	5	$1.60604, -1.0472, -1.60604, 1.60604, -1.0472, -1.60604$
24	19	5	$3.01302, 2.55645, -2.51206, -1.25786, -2.13757, -0.499749$
25	19	5	$-1.25786, -2.13757, -0.499749, 3.01302, 2.55645, -2.51206$
26	31	5	$1.34557, 0.428593, -1.39623, -1.11705, 1.24692, 2.42435$
27	31	5	$-1.11705, 1.24692, 2.42435, 1.34557, 0.428593, -1.39623$

TABLE 2. $(\omega_j, \tau_j, \xi_j)$ for J_6^1 in Table 1

Modular Data when $G = \mathbb{Z}/6\mathbb{Z}$

- The center $\mathcal{C}$ has rank 54 with the following simple objects:
  - 6 invertible objects
  - 6 with dimension $4 + \sqrt{15}$
  - 15 with dimension $5 + \sqrt{15}$
  - 27 with dimension $3 + \sqrt{15}$
- S and T for its pointed subcategory:

$$S_{\text{pt}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} & 1 & e^{-\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} \\ 1 & e^{\frac{2i\pi}{3}} & e^{-\frac{2i\pi}{3}} & 1 & e^{\frac{2i\pi}{3}} & e^{-\frac{2i\pi}{3}} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} & 1 & e^{-\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} \\ 1 & e^{\frac{2i\pi}{3}} & e^{-\frac{2i\pi}{3}} & 1 & e^{\frac{2i\pi}{3}} & e^{-\frac{2i\pi}{3}} \end{bmatrix},$$

$$T_{\text{pt}} = \text{diag}\left[1, e^{\frac{i\pi}{3}}, e^{-\frac{2i\pi}{3}}, -1, e^{-\frac{2i\pi}{3}}, e^{\frac{i\pi}{3}}\right].$$

- $\mathcal{C}$ contains a fermion f.
- $\mathcal{C}$ has a pointed modular subcategory $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)$.

  $\mathcal{C} \cong \mathcal{D} \boxtimes \mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)$, where $\mathcal{D}$ is a spin modular category.

Realizing the 1st Modular Data

- The 1st family of modular data can be obtained from the center of near-group categories of type $\mathbb{Z}/6\mathbb{Z}+6$.

Theorem (Rowell-Solomon-Z)

Let $\mathcal{C}$ be the Drinfeld center of a near-group category of type $\mathbb{Z}/6+6$, then $\mathcal{C} \cong \mathcal{D} \boxtimes \mathcal{C}(\mathbb{Z}/3, q)$, where $\mathcal{D}$ is a spin modular category. Moreover, the Müger centralizer of $\langle f \rangle$ in $\mathcal{D}$ is super-modular and has the following family of modular data

$$\hat{S} = \begin{bmatrix} 1 & 4+\sqrt{15} & 5+\sqrt{15} & 3+\sqrt{15} & 3+\sqrt{15} \\ 4+\sqrt{15} & 1 & 5+\sqrt{15} & -3-\sqrt{15} & -3-\sqrt{15} \\ 5+\sqrt{15} & 5+\sqrt{15} & -5-\sqrt{15} & 0 & 0 \\ 3+\sqrt{15} & -3-\sqrt{15} & 0 & \frac{1}{2}(1+\sqrt{5})(3+\sqrt{15}) & -\frac{2\sqrt{30}(4+\sqrt{15})}{5+\sqrt{5}} \\ 3+\sqrt{15} & -3-\sqrt{15} & 0 & -\frac{2\sqrt{30(4+\sqrt{15})}}{5+\sqrt{5}} & \frac{1}{2}(1+\sqrt{5})(3+\sqrt{15}) \end{bmatrix}$$

$$\hat{T}^2 = \mathrm{Diag}\left[1, 1, e^{2i\pi/3}, e^{4i\pi/5}, e^{-4i\pi/5}\right]$$

Modular Data when $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

- [Evans-Gannon '14]
  There are 4 near-group categories when $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

	c	$\langle\ ,\ \rangle$	a	$b(0,1), b(0,2), b(1,0), b(1,1), b(1,2)$
$J^1_{(2,4)}$	ζ_{12}^{5}	1	$(1,\ 1)$	$-0.992441, 1.5708, 0.785398, -1.42977, -0.785398$
$J^2_{(2,4)}$	ζ_{12}^{-5}	-1	$(-1,\ 1)$	$0.992441, -1.5708, -0.785398, 1.42977, 0.785398$
$J^3_{(2,4)}$	ζ_{12}^{-5}	1	$(1,\ -1)$	$1.42977, -1.5708, 0.785398, -1.77784, -0.785398$
$J^4_{(2,4)}$	ζ_{12}^{5}	-1	$(-1,\ -1)$	$-1.42977, 1.5708, -0.785398, 1.77784, 0.785398$

- There are 44 solutions to the triples (ω, ξ, τ).

#	ω	τ	ξ
1	16	$(0,0)$	$0, 0.4373, 0, -2.008, -1.571, 3.076, 1.571, -1.505$
2	16	$(0,0)$	$0, -2.008, 0, 0.4373, 1.571, -1.505, -1.571, 3.076$
3	40	$(0,0)$	$1.571, -1.505, -1.571, 3.076, 0, -2.008, 0, 0.4373$
4	40	$(0,0)$	$-1.571, 3.076, 1.571, -1.505, 0, 0.4373, 0, -2.008$
5	21	$(0,1)$	$-1.134, 1.003, -0.0339, 3.045, -0.0339, 3.045, -1.134, 1.003$
6	45	$(0,1)$	$-0.0339, 3.045, -1.134, 1.003, -1.134, 1.003, -0.0339, 3.045$
7	13	$(0,1)$	$0.2202, -1.398, -1.675, -2.644, -2.842, -1.477, 2.793, 2.312$
8	13	$(0,1)$	$2.793, 2.312, -2.842, -1.477, -1.675, -2.644, 0.2202, -1.398$
9	37	$(0,1)$	$-1.675, -2.644, 0.2202, -1.398, 2.793, 2.312, -2.842, -1.477$
10	37	$(0,1)$	$-2.842, -1.477, 2.793, 2.312, 0.2202, -1.398, -1.675, -2.644$
11	4	$(0,2)$	$3.076, 1.571, -1.505, -1.571, 0.4373, 0, -2.008, 0$
12	4	$(0,2)$	$-1.505, -1.571, 3.076, 1.571, -2.008, 0, 0.4373, 0$
13	28	$(0,2)$	$0.4373, 0, -2.008, 0, 3.076, 1.571, -1.505, -1.571$
14	28	$(0,2)$	$-2.008, 0, 0.4373, 0, -1.505, -1.571, 3.076, 1.571$
15	21	$(0,3)$	$1.003, -0.0339, 3.045, -1.134, 3.045, -1.134, 1.003, -0.0339$
16	45	$(0,3)$	$3.045, -1.134, 1.003, -0.0339, 1.003, -0.0339, 3.045, -1.134$
17	13	$(0,3)$	$2.312, -2.842, -1.477, 2.793, -2.644, 0.2202, -1.398, -1.675$
18	13	$(0,3)$	$-1.398, -1.675, -2.644, 0.2202, -1.477, 2.793, 2.312, -2.842$
19	37	$(0,3)$	$-2.644, 0.2202, -1.398, -1.675, 2.312, -2.842, -1.477, 2.793$
20	37	$(0,3)$	$-1.477, 2.793, 2.312, -2.842, -1.398, -1.675, -2.644, 0.2202$
21	22	$(1,0)$	$-2.241, -1.335, -0.835, 0.8973, 1.455, 3.03, 0.04963, -1.022$
22	22	$(1,0)$	$-0.835, 0.8973, -2.241, -1.335, 0.04963, -1.022, 1.455, 3.030$
23	22	$(1,0)$	$0.04963, -1.022, 1.455, 3.03, -0.835, 0.8973, -2.241, -1.335$
24	22	$(1,0)$	$1.455, 3.03, 0.04963, -1.022, -2.241, -1.335, -0.835, 0.8973$
25	6	$(1,0)$	$2.235, 2.673, 2.235, 2.673, 1.168, -0.8402, 1.168, -0.8402$
26	6	$(1,0)$	$1.168, -0.8402, 1.168, -0.8402, 2.235, 2.673, 2.235, 2.673$
27	15	$(1,1)$	$0.6315, -3.119, 2.979, -2.324, 0.6315, -3.119, 2.979, -2.324$
28	39	$(1,1)$	$2.979, -2.324, 0.6315, -3.119, 2.979, -2.324, 0.6315, -3.119$
29	7	$(1,1)$	$2.65, 1.091, 2.736, -0.06957, 1.658, 0.09927, -0.3231, -3.129$
30	7	$(1,1)$	$1.658, 0.09927, -0.3231, -3.129, 2.65, 1.091, 2.736, -0.06957$
31	31	$(1,1)$	$2.736, -0.06957, 2.65, 1.091, -0.3231, -3.129, 1.658, 0.09927$
32	31	$(1,1)$	$-0.3231, -3.129, 1.658, 0.09927, 2.736, -0.06957, 2.65, 1.091$
33	34	$(1,2)$	$-1.022, 1.455, 3.03, 0.04963, 0.8973, -2.241, -1.335, -0.835$
34	34	$(1,2)$	$0.8973, -2.241, -1.335, -0.835, -1.022, 1.455, 3.03, 0.04963$
35	34	$(1,2)$	$3.03, 0.04963, -1.022, 1.455, -1.335, -0.835, 0.8973, -2.241$
36	34	$(1,2)$	$-1.335, -0.835, 0.8973, -2.241, 3.03, 0.04963, -1.022, 1.455$
37	18	$(1,2)$	$2.673, 2.235, 2.673, 2.235, -0.8402, 1.168, -0.8402, 1.168$
38	18	$(1,2)$	$-0.8402, 1.168, -0.8402, 1.168, 2.673, 2.235, 2.673, 2.235$
39	15	$(1,3)$	$-3.119, 2.979, -2.324, 0.6315, -3.119, 2.979, -2.324, 0.6315$

Modular Data when $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

- The center $\mathcal{C}$ has rank 88 with the following simple objects:
  - 8 invertible objects
  - 8 with dimension $5 + 2\sqrt{6}$
  - 28 with dimension $2(3 + 2\sqrt{6})$
  - 44 with dimension $4 + 2\sqrt{6}$

- S and T for its pointed subcategory:

$$S_{\mathrm{pt}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

$$T_{\mathrm{pt}} = \mathrm{diag}\,[1,\, i,\, 1,\, i,\, -1,\, -i,\, -1,\, -i].$$

- Boson: self-dual invertible object with twist $\theta_b = 1$.

- $\mathcal{C}$ has two fermions and one boson b.

Boson Condensation

- Let $\mathcal{C}$ be modular and $\mathrm{Rep}(G) \cong \mathcal{D} \subset \mathcal{C}$ a Tannakian subcategory.
- The G-de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ is a braided G-crossed category.
- Trivial component $[\mathcal{C}_G]_e$ is modular, $\dim([\mathcal{C}_G]_e) = \dim(\mathcal{C})/|G|^2$
- $[\mathcal{C}_G]_e$ is the boson condensation of $\mathcal{C}$, $[\mathcal{C}_G]_e = (C_{\mathcal{C}}(\mathcal{D}))_G$

If $\mathrm{Rep}\,(\mathbb{Z}_2) \cong \langle b \rangle \subset \mathcal{C}$, let F be the $\mathbb{Z}/2$-de-equivariantization functor.

Restricting to the centralizer of $\langle b \rangle$,

- if $b \otimes X \cong Y \neq X$, then $F(X) \cong F(Y)$ is simple with same dimension.
- if $b \otimes X \cong X$, then $F(X) \cong X_1 \oplus X_2$, X_i simple with $\dim(X)/2$.
- $\theta_{F(X)} = \theta_X$.

From the Center to Super-modular

Let $\mathcal{C}$ be the Drinfeld center of a near-group category of type $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} + 8$.

- Condense the boson b to obtain modular category $[\mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}]_0$.
  - rank $= 36$.
  - dimensions are 1, $5 + 2\sqrt{6}$, $2(3 + \sqrt{6})$, $3 + \sqrt{6}$, $4 + 2\sqrt{6}$, $2 + \sqrt{6}$.
  - $\left[\mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}\right]_0$ contains semion objects (self-dual invertible with twist $\pm i$).
- $[\mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}]_0 \cong \mathcal{D} \boxtimes \mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q)$, where $\mathcal{D}$ is a spin modular category.
- $\mathcal{D} \cong \mathcal{D}_0 \oplus \mathcal{D}_1$, $\mathcal{D}_0 = C_{\mathcal{C}}(\langle f \rangle)$.
- $\mathcal{D}_0$ is a super-modular category.

Realizing the 2nd Class of Modular Data

- The 2nd class of modular data can be obtained from the center of near-group categories of type $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} + 8$.

Theorem (Rowell-Solomon-Z)

Let $\mathcal{C}$ be the Drinfeld center of a near-group category of type $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} + 8$. Then $[\mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}]_0 \cong \mathcal{D} \boxtimes \mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q)$, where $\mathcal{D}$ is a spin modular and q is the associated quadratic form restrict to $\mathbb{Z}/2\mathbb{Z}$. The Müger centralizer of $\langle f \rangle$ in $\mathcal{D}$ has the following family of modular data

$$\hat{S} = \begin{bmatrix} 1 & 5+2\sqrt{6} & 3+\sqrt{6} & 3+\sqrt{6} & 4+2\sqrt{6} \\ 5+2\sqrt{6} & 1 & 3+\sqrt{6} & 3+\sqrt{6} & -4-2\sqrt{6} \\ 3+\sqrt{6} & 3+\sqrt{6} & -3-\sqrt{6}-i\sqrt{6\left(2\sqrt{6}+5\right)} & -3-\sqrt{6}+i\sqrt{6\left(2\sqrt{6}+5\right)} & 0 \\ 3+\sqrt{6} & 3+\sqrt{6} & -3-\sqrt{6}+i\sqrt{6\left(2\sqrt{6}+5\right)} & -\sqrt{6}-i\sqrt{6\left(2\sqrt{6}+5\right)}-3 & 0 \\ 4+2\sqrt{6} & -4-2\sqrt{6} & 0 & 0 & 4+2\sqrt{6} \end{bmatrix}$$

$$\hat{T}^2 = \mathrm{Diag}\left[1, 1, -1, -1, e^{2i\pi/3}\right]$$

Other Examples from Near-group Centers

Type of $\mathcal{F}$	Conj. form of $\mathcal{Z}(\mathcal{F})$	Notes
$\mathbb{Z}/1+1$	$\mathrm{Fib}\boxtimes\mathrm{Fib}^{\mathrm{rev}}$	$\mathcal{F}=\mathrm{Fib}$
$\mathbb{Z}/2+2$	$\mathrm{SU}(2)_{10}$	[BRW]
$\mathbb{Z}/3+3$	$\mathrm{G}(2)_3\boxtimes\mathcal{C}(\mathbb{Z}/3,Q)$	$\mathrm{rank}(\mathrm{G}(2)_3)=6$ [NRWW]
$\mathbb{Z}/2\times\mathbb{Z}/2+4$	$\left(\left[\left[\mathrm{SU}(2)_6^{\boxtimes 2}\right]_{\mathbb{Z}/2}\right]_0\right)_{\mathbb{Z}/2}^{\times,\mathbb{Z}/2}$	[BPRZ]
$\mathbb{Z}/4+4$	$[\mathrm{PSU}(3)_5\boxtimes\mathcal{C}(\mathbb{Z}/2,Q)]_{\mathbb{Z}/2}^{\times,\mathbb{Z}/2}$	$\mathrm{rank}(\mathrm{PSU}(3)_5)=7$
$\mathbb{Z}/5+5$	$\mathcal{B}\boxtimes\mathcal{C}(\mathbb{Z}/5,Q)$	$\mathrm{rank}(\mathcal{B})=8$
$\mathbb{Z}/6+6$	$\mathcal{D}\boxtimes\mathcal{C}(\mathbb{Z}/3,Q)$	$\mathcal{D}$ in Theorem 1 [RSZ]
$\mathbb{Z}/7+7$	$\mathcal{B}\boxtimes\mathcal{C}(\mathbb{Z}/7,Q)$	$\mathrm{rank}(\mathcal{B})=10$
$\mathbb{Z}/8+8$	$[G(2)_4\boxtimes\mathcal{C}(\mathbb{Z}/4,Q)]_{\mathbb{Z}/2}^{\times,\mathbb{Z}/2}$	$\mathrm{rank}(G(2)_4)=9$
$\mathbb{Z}/2\times\mathbb{Z}/4+8$	$[\mathcal{D}\boxtimes\mathcal{C}(\mathbb{Z}/2,Q)]_{\mathbb{Z}/2}^{\times,\mathbb{Z}/2}$	$\mathcal{D}$ in Theorem 2 [RSZ]

Table: Familiar categories conjecturally related to centers of $G+n$ near-group categories.

Thank you!