Modular Data of Non-semisimple Modular Categories

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Related work

- [Lyubashenko95] Projective $\mathrm{SL}_2(\mathbb{Z})$-action on a Hopf algebra in a (non-semisimple) modular category over $\mathbb{C}$.

- [Lyubashenko-Majid94] Projective $\mathrm{SL}_2(\mathbb{Z})$-action on a finite dimensional factorizable ribbon Hopf algebra H:

$$S_{LM}(x) = (\mathrm{id} \otimes \lambda)\left(R_{21}(1 \otimes x)R\right), \quad T_{LM}(x) = \theta x$$

  for all $x \in H$.

  - $S_{LM} = \mathbf{f}_Q \circ \Psi$.
  - $(S_{LM})^4 = S^2$, where S is the antipode.
  - The center $Z(H)$ is S_{LM}-stable.

Related work

- [Kerler95] Let q be primitive $2r+1$ th root. The $\mathrm{SL}_2(\mathbb{Z})$-representation on the center of $u_q\mathfrak{sl}_2$ decomposes as

$$Z\left(u_q\mathfrak{sl}_2\right) = \mathcal{P}_{r+1} \oplus \mathbb{C}^2 \otimes \mathcal{V}_r.$$

- [Lachowska03] An ideal of the center of small quantum group which is $\mathrm{SL}_2(\mathbb{Z})$-invariant.

- [Cohen-Westreich08] The Higman ideal of H is a $\mathrm{SL}_2(\mathbb{Z})$-submodule of $Z(H)$; a Verlinde-type formula for $\mathrm{Hig}(H)$.

- [Gainutdinov-Runkel20] Generalize the results to modular categories.

Modular category

Let $\mathcal{C}$ be a braided tensor category over $\mathbb{C}$.

- Müger center of $\mathcal{C}$:

$$\mathrm{Ob}\,(\mathcal{C}') := \{X \in \mathcal{C} \mid c_{Y,X} c_{X,Y} = \mathrm{Id}_{X \otimes Y}\,,\ \forall\ Y \in \mathcal{C}\}$$

- $\mathcal{C}$ is symmetric if $\mathcal{C}' \cong \mathcal{C}$.

- $\mathcal{C}$ is non-degenerate if $\mathcal{C}' \cong \mathrm{Vec}$.

- A modular category $\mathcal{B}$ is a non-degenerate ribbon finite tensor category.

From Hopf algebra to modular category

Let H be a Hopf algebra.

- (H, R) is quasi-triangular. $\iff$ Rep(H) is braided.
- (H, R) is a ribbon algebra $\iff$ Rep(H) is a ribbon category.

  (H, R) is a ribbon algebra if $\exists\, \theta$ (central) $\in H$ s.t.
  $\Delta(\theta) = (R_{21}R_{12})^{-1}(\theta \otimes \theta), \quad \epsilon(\theta) = 1, \quad$ and $\quad S(\theta) = \theta$

- (H, R) is factorizable $\iff$ Rep(H) is non-degenerate.

  (H, R) is called factorizable if the $\mathbf{f}_Q$ is an isomorphism of vector spaces, where $\mathbf{f}_Q$ is the Drinfeld map $\mathbf{f}_Q : H^* \to H$ defined by

  $$\mathbf{f}_Q(p) = (p \otimes \mathrm{Id})(Q)$$

  where $Q = R_{21}R_{12}$.

Modular data of semisimple modular categories

- Verlinde formula

$$N_{XY}^{Z} = \sum_{W} \frac{S_{XW} S_{YW} S_{Z^* W}}{S_{\mathbf{1} W}}$$

- Congruence kernel of the $\mathrm{SL}_2(\mathbb{Z})$-representation ρ
  - $H < \mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if H contains some $\Gamma(n) := \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv I \pmod{n}\}$.
  - $\mathrm{SL}_2(\mathbb{Z})/\Gamma(n) \cong \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for all $n > 1$.

[Ng-Schauenburg '10] If $N = \mathrm{ord}(T)$,

- ρ factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.
- $\mathbb{Q}(S) \subset \mathbb{Q}(\zeta_N)$, $N = \mathrm{ord}(T)$.

Non-semisimple modular categories

Let $\mathcal{C}$ be a non-semisimple modular category.

- Total rank: rank $+ |\{$proj. indecomp. objects not simple$\}|$
- Steinberg object: An object is called Steinberg if it is both simple and projective.
- [Cohen-Westreich08] Let H be a factorizable ribbon Hopf algebra. Then there exists at least one simple and projective H-module.
- [Gainutdinov-Runkel20] $\mathcal{C}$ always contains a Steinberg object.

Proposition [Chang-Kolt-Wang-Z]

Any modular category of total rank < 5 is semisimple.

Non-semisimple modular categories from $u_q\mathfrak{sl}_2$

Let $q \in \mathbb{C}^\times, q \neq \pm 1$.

Quantum group $U_q\mathfrak{sl}_2$

$U_q\mathfrak{sl}_2 = \mathbb{C}\left\langle E, F, K^{\pm 1} \middle| [E,F] = \frac{K-K^{-1}}{q-q^{-1}}, KE = q^2 EK, KF = q^{-2} FK \right\rangle$

$\Delta(E) = 1 \otimes E + E \otimes K$, $\Delta(F) = K^{-1} \otimes F + F \otimes 1$, $\Delta(K^\pm) = K^\pm \otimes K^\pm$

$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = \epsilon\left(K^{-1}\right) = 1.$

$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S\left(K^{-1}\right) = K.$

Small quantum group $u_q\mathfrak{sl}_2$

Let q be a primitive l th root of unity, where l is odd.

$u_q\mathfrak{sl}_2 = U_q\mathfrak{sl}_2/(E^l, F^l, K^l - 1)$.

$u_q\mathfrak{sl}_2$ is finite dimensional with basis $\left\{E^i F^j K^\ell\right\}_{0 \leq i,j,\ell \leq p-1}$.

$\mathrm{Rep}(u_q\mathfrak{sl}_2)$ is a finite tensor category.

Non-semisimple modular categories from $u_q\mathfrak{sl}_2$ (continued)

Let q be a primitive odd $l = 2h + 1 \geq 3$ th root of unity. The Hopf algebra $u_q\mathfrak{sl}_2$ is

- Quasi-triangular

$$R = \frac{1}{l} \sum_{0 \leq i,j,k \leq l-1} \frac{\left(q - q^{-1}\right)^k}{[k]!} q^{k(k-1)/2+2k(i-j)-2ij} E^k K^i \otimes F^k K^j$$

- Ribbon $\theta =$
$$\frac{1}{l} \left(\sum_{r=0}^{l-1} q^{hr^2} \right) \left(\sum_{0 \leqslant m,j \leqslant l-1} \frac{\left(q - q^{-1}\right)^m}{[m]!} (-1)^m q^{-\frac{1}{2}m+mj+\frac{1}{2}(j+1)^2} F^m E^m K^j \right)$$

- Factorizable

- Therefore, $\mathrm{Rep}(u_q\mathfrak{sl}_2)$ is a non-semisimple modular category.

Double of Nichols Hopf algebra

Nichols Hopf Algebra

$\mathcal{K}_n = \mathbb{C}\left\langle K, \xi_i, i = 1, 2, \ldots, n \mid K\xi_i K = -\xi_i, \xi_i\xi_j = -\xi_j\xi_i, K^2 = 1 \right\rangle,$
$\Delta(K) = K \otimes K,\ \Delta(\xi_i) = K \otimes \xi_i + \xi_i \otimes 1,$
$\epsilon(K) = 1,\ \epsilon(\xi_i) = 0.$
$S(K) = K,\ S(\xi_i) = -K\xi_i.$

- $\mathcal{K}_n$ is a Hopf algebra of dimension 2^{n+1}.
- $D\mathcal{K}_n$ is factorizable.
- $D\mathcal{K}_n$ is ribbon when n is even.
- Rep($D\mathcal{K}_n$) is modular when n is even.
  - Total Rank: 6 for all n.

Cartan Matrix

Let $\mathcal{C}$ be finite tensor category.

- A filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$ is called a Jordan-Hölder series if X_i/X_{i-1} is simple for all i.
- By the Jordan-Hölder theorem, the number of any simple object $Y = X_i/X_{i-1}$ is the same in any two Jordan-Hölder series.

  Denote this number by $[X : Y]$ and call it the multiplicity of Y in X.
- The Cartan matrix C of $\mathcal{C}$ has entries

  $$C_{i,j} = [P_i : V_j],$$

  where V_i, P_i are simple objects in $\mathcal{C}$ and their projective covers.
- [Etingof-Ostrik04] $\det(C) = 0$ if $\mathcal{C}$ is non-semisimple and admits an isomorphism of additive functors $u : \mathrm{Id}_{\mathcal{C}} \to **$.
- Two projective indecomposable objects X, Y are c-equivalent if they have the same column in the Cartan matrix.

Fusion rings - $\mathrm{Gr}(\mathcal{C})$

Let $\mathcal{C}$ be finite tensor category.

$\mathrm{Gr}(\mathcal{C})$

- The associated class $[X] \in \mathrm{Gr}(\mathcal{C})$ of $X \in \mathcal{C}$ is given by
$$[X] = \sum_{X_i \in \mathrm{Irr}(\mathcal{C})} [X : X_i] \, X_i .$$
Multiplication: $[X_j]\,[X_k] := [X_j \otimes X_k]$

- Projective objects become sums of simple objects.

  E.g., $\mathrm{Rep}\,(u_q\mathfrak{sl}_2)$, q is a primitive 3rd root
  Simple objects V_1, V_2, V_3.
  Projective covers $P_1, P_2, P_3 \cong V_3$.
  $[P_1] = 2\,[V_1] + 2\,[V_2]$.

Fusion rings - $K_0(\mathcal{C})$

$K_0(\mathcal{C})$

- Abelian group generated by the isomorphism classes $[P]$ of projective object P modulo the relations $[P \oplus Q] = [P] + [Q]$.

  Multiplication:$[P][Q] = [P \otimes Q]$ for any projective objects P and Q in $\mathcal{C}$.

- No elements in $K_0(\mathcal{C})$ representing simple V_i unless it is projective.

- In general, $K_0(\mathcal{C})$ can be a ring without a unit (rng).

Mixed fusion module

- Consider $\{[P_i]_c\}_{V_i \in \mathrm{Irr}(\mathcal{C})}$, where $[P_i]_c$ is the c-class of P_i.
- $V_i \otimes P_j$ decomposes as a direct sum of indecomposable projective objects.
- Mixed fusion coefficients

$$V_i \otimes P_j \cong \oplus_{k=1}^{m} N_{ij}^{k}(m)\,[P_k]_c\,,$$

  where
  - m is the rank of the Cartan matrix,
  - $N_{ij}^k(m) = \sum_{P_k \in [P_k]_c} N_{ij}^k(t)$, where $N_{ij}^k(t)$ is the fusion coefficient in the full fusion ring of $\mathcal{C}$.
- Mixed fusion matrices: $N^i(m)$, where $\left(N^i(m)\right)_{jk} = N_{ij}^k(m)$.

Examples of low-rank NSS modular categories

Consider $\mathrm{Rep}(u_q\mathfrak{sl}_2)$, where q is a 3rd root:

- Total rank 5:

  Simple objects V_1, V_2, V_3
  Projective covers P_1, P_2, $P_3 \cong V_3$ (Steinberg object)

- Cartan matrix:

$$C = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  P_1 and P_2 are c-equivalent.

- Fusion rules:

  $V_1 \otimes X \cong X$, $V_2 \otimes V_2 \cong V_1 \oplus V_3$, $V_2 \otimes V_3 \cong P_2$, $V_2 \otimes P_1 \cong 2V_3 \oplus P_2$,
  $V_2 \otimes P_2 \cong 2V_3 \oplus P_1$, $V_3 \otimes V_3 \cong V_3 \oplus P_1$, $V_3 \otimes P_1 \cong 2V_3 \oplus 2P_2$,
  $V_3 \otimes P_2 \cong 2V_3 \oplus 2P_2$.

Examples of low-rank NSS modular categories (continued)

Consider $\text{Rep}\,(D\mathcal{K}_n)$, where n is an even number.

- Total rank 6:

  Simple objects: $V_1, V_{K\bar{K}}, V_K, V_{\bar{K}}$
  Projective covers: $P_1, P_{K\bar{K}}$, $V_K, V_{\bar{K}}$ (Steinberg objects)

- Cartan matrix:

$$C = \begin{pmatrix} 2^{2n-1} & 2^{2n-1} & 0 & 0 \\ 2^{2n-1} & 2^{2n-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  P_1 and $P_{K\bar{K}}$ are c-equivalent.

- Does finiteness hold for modular categories with the same Cartan matrix?

- Mixed fusion matrices:

  $N^1 = I_3$, $N^K = N^{\bar{K}} = \begin{pmatrix} 0 & 1 & 1 \\ 2^{2n-1} & 0 & 0 \\ 2^{2n-1} & 0 & 0 \end{pmatrix}$, $N^{K\bar{K}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Higman ideal of a finite dimensional Hopf algebra

Let H be a f.d. Hopf algebra.

- H is a left H-module via the left adjoint action:

$$(\operatorname{ad} h)(a) := \sum h_1 a S(h_2)$$

  for all $h, a \in H$ and S is the antipode of H.

- Let $\Lambda \in H$ be the left integral of H.
  Then $(\operatorname{ad} \Lambda)(H)$ is an ideal of $Z(H)$.

- Define the Higman ideal of H by

$$\operatorname{Hig}(H) = (\operatorname{ad} \Lambda)(H)$$

- [Lorenz97] $\dim(\operatorname{Hig}(H)) = \operatorname{rank}(C)$, where C is the Cartan matrix.

Higman ideal of a finite dimensional Hopf algebra

Let H be a f.d. factorizable ribbon Hopf algebra.

- Define $p_a \in H^*$ by

$$\langle p_a, b \rangle := \mathrm{Tr}(l(b) \circ r(a)),$$

  for $b \in H$, where $l(b)$, $r(a)$ denote left and right multiplication by a and b.
- Let $\{V_i\}_{i=1}^n$ be the set of irre. left H-module and $\{e_i\}_{i=1}^n$ be the corresponding orthogonal primitive idempotent.
- $p_{e_i} = \chi_{Ae_i} = \sum_{k=1}^{n} c_{ki}\chi_k$, where $\chi_k = \chi_{V_k}$ is the irre. character.
- [Bass76] Let $I(H) = \{p_a \mid a \in H\}$, then

$$I(H) = \mathrm{Sp}_{\mathbb{k}}\{p_e | \, e \text{ a primitive idempotent of } H\}$$

- [Cohen-Westreich08] $\dim(I(H)) =$ rank of the Cartan matrix.

Cohen-Westreich S-matrix

Let (H, R) be a f.d. factorizable ribbon Hopf algebra

- Let $u = \sum_i S(t_i)\, s_i$, $R = \sum s_i \otimes t_i \in H \otimes H$.
- Let $f_Q : H^* \to H$ be the Drinfeld map, θ be the ribbon element, and λ be the right integral of H^*.
- Consider the $n \times n$ matrix

$$\begin{aligned} B_{i,j} &= \left\langle \widehat{f_Q}\left(p_{e_i}\right), \widehat{\Psi} S\left(e_j\right) \right\rangle \\ &= \left\langle f_Q\left(\left(\sum_{k=1}^{n} c_{ki} \chi_k \right) \leftharpoonup u^{-1}\theta \right), \left(\lambda \leftharpoonup \theta^{-1} u e_j\right) \right\rangle \end{aligned}$$

  then B is a symmetric matrix.
- For any matrix A we denote by A_m the $m \times m$ major minor of A.
- Define $S_{CW} = C_m^{-1} B_m$, where C is the Cartan matrix of rank m.

Cohen-Westreich S-matrix (continued)

- S_{CW} is the change of basis matrix for the following two bases for the Higman ideal

$$B_\chi = \left\{ \widehat{f_Q}\left(p_{e_j}\right) \right\}_{j=1}^{m}, \quad B_\tau = \left\{ (\mathrm{ad}\,\Lambda)\left(e_j\right) \right\}_{j=1}^{m}$$

- For all $1 \leqslant i \leqslant n$,

$$S_{CW}^{-1} N^i(m) S_{CW} = \mathrm{Diag}\left\{ d_1^{-1} s_{i1}, \ldots, d_m^{-1} s_{im} \right\}$$

where $d_i = \dim\left(V_i\right)$, $s_{ij} = \left\langle \widehat{f_Q}\left(\chi_i\right), \chi_j \right\rangle$.

- [Gainutdinov-Runkel20] Generalizes to modular categories.

Center of $u_q\mathfrak{sl}_2$

[Kerler95] Let $l = 2r + 1$ be an odd number and q be a primitive lth root of unity. The $\mathrm{SL}_2(\mathbb{Z})$ representation on the center Z of $u_q\mathfrak{sl}_2$ decomposes as

$$Z = \mathcal{P}_{r+1} \oplus \mathbb{C}^2 \otimes \mathcal{V}_r,$$

where

- $\mathcal{P}_{r+1}$ is an $(r+1)$-dimensional representation.
- $\mathbb{C}^2$ is the standard representation of $\mathrm{SL}_2(\mathbb{Z})$.
- $\mathcal{V}_r$ is an r dimensional representation when restricted on which the matrices S_{LM} and T_{LM} are the same as those obtained by a semisimple modular category.

Cohen-Westreich modular data for $u_q\mathfrak{sl}_2$

- Restricting to $\mathcal{P}_{r+1}$,

$$S_{CW} = \frac{1}{\sqrt{l}} \begin{pmatrix} 1 & 1 & & \cdots & & 1 \\ 2 & q+q^{-1} & & \cdots & & q^r+q^{-r} \\ & & \ddots & & & \\ \vdots & \vdots & & q^{js}+q^{-js} & & \vdots \\ & & & & \ddots & \\ 2 & q^r+q^{-r} & & \cdots & & q^{r^2}+q^{-r^2} \end{pmatrix}_{(r+1)\times(r+1)}$$

$$T_{CW} = \kappa\, \mathrm{Diag}\{-q^{-\frac{1}{2}}, 1, \cdots, (-1)^{s+1} q^{\frac{1}{2}(r^2-1)}, \cdots, (-1)^{r+1} q^{\frac{1}{2}(r^2-1)}\}$$

- Restricting to $\mathcal{V}_r$, (S_{semi}, T_{semi}) is a $\mathrm{PSU}(2)_{l-2}$ up to a Galois conjugation.
- (S_{CW}, T_{CW}), (S_{semi}, T_{semi}) corresponds to the even and odd part of the modular data from a pointed modular tensor category $\mathcal{C}(\mathbb{Z}/l\mathbb{Z}, Q)$, respectively.

Theorem [Chang-Kolt-Wang-Z]

The $\mathrm{SL}_2(\mathbb{Z})$-representation on the Higman ideal of $u_q\mathfrak{sl}(2)$ has kernel congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Cohen-Westreich modular data for $D\mathcal{K}_n$

Suppose n is even.

- The S_{CW} - and T_{CW}-matrices on the Higman ideal of $D\mathcal{K}_n$ are given by

$$S_{CW} = (-1)^{n/2} \begin{pmatrix} 0 & 2^{-n} & -2^{-n} \\ 2^{n-1} & 1/2 & 1/2 \\ -2^{n-1} & 1/2 & 1/2 \end{pmatrix}, \quad T_{CW} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

- This can be further decomposed as $\mathbb{C}_{\text{triv}} \oplus N_1$, where N_1 is the level 2 representation.

Corollary [Chang-Kolt-Wang-Z]

The $\mathrm{SL}_2(\mathbb{Z})$-representation on the Higman ideal of $D\mathcal{K}_n$ has kernel congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Conjecture

The congruence kernel theorem holds for the Cohen-Westreich modular data of a non-semisimple modular category.

Thank You!