Lecture 2 Integrals as General and Particular Solutions

Integrating Both Sides

- The first-order equation $\frac{dy}{dx} = f(x, y)$ takes an especially simple form if the right-hand-side function f does not actually involve the dependent variable y, so

$$y' = \frac{dy}{dx} = f(x) \tag{1}$$

- In this special case we need only integrate both sides of the equation to obtain

$$y(x) = \int f(x)dx + C \tag{2}$$

- This is a general solution of the differential equation, meaning that it involves an arbitrary constant C, and for every choice of C it is a solution of the differential equation.

Example 1 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

Initial condition

$$\frac{dy}{dx} = (x-2)^2; \; \underline{y(2) = 1}$$

Recall

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

ANS: We have $\frac{dy}{dx} = x^2 - 4x + 4$

Integrating both sides, we have

$$y = \int x^2 - 4x + 4 \, dx + C$$

$$= \frac{1}{3}x^3 - 2x^2 + 4x + C$$

general solution

Thus $\underline{y(x) = \frac{1}{3}x^3 - 2x^2 + 4x + C, \text{ where } C \text{ is any constant.}}$

As $y(2) = 1$, we have

$$y(2) = \frac{1}{3} \cdot 2^3 - 2 \cdot 4 + 4 \cdot 2 + C = 1. \Rightarrow C = 1 - \frac{8}{3} = -\frac{5}{3}$$

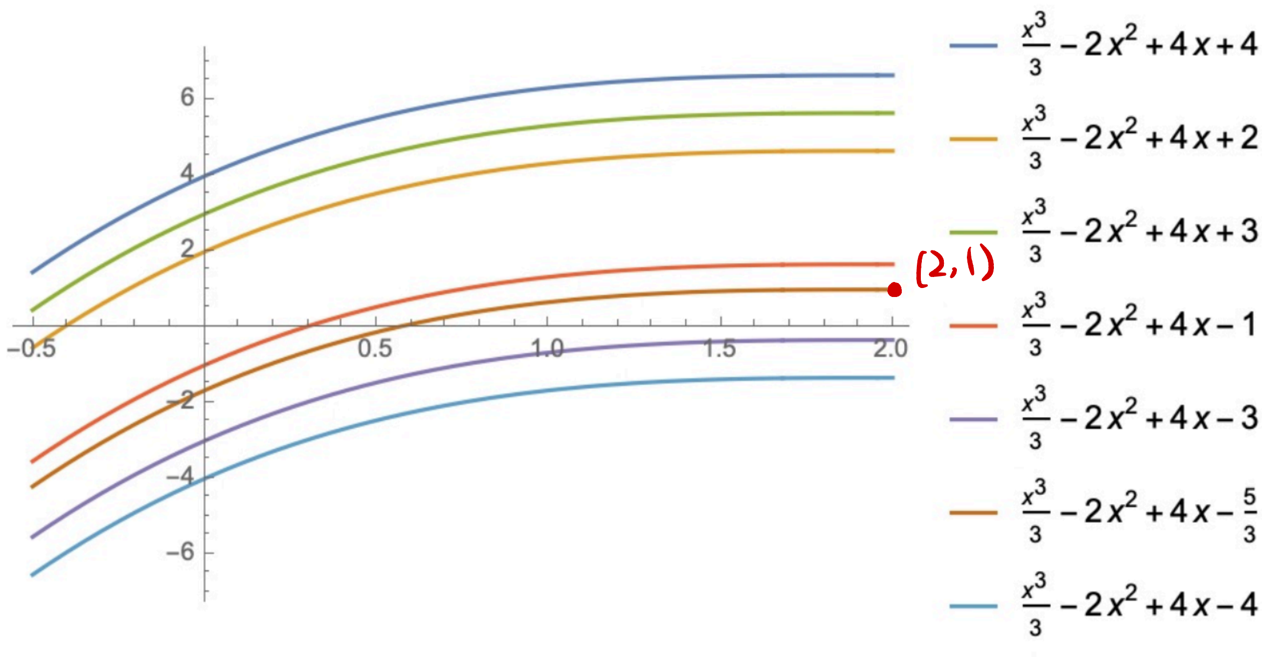
Thus $y(x) = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{5}{3}$. (particular solution)

Let's look at the graph of the general solution

$y = \frac{1}{3}x^3 - 2x^2 + 4x + C$

and the particular solution

$y = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{5}{3}$



Example 2 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

$$\frac{dy}{dx} = \frac{1}{\sqrt{x+4}};\ y(5) = 7$$

ANS: Integrate both sides.

$$y = \int \frac{1}{\sqrt{x+4}}\,dx + c$$

$$= \int (x+4)^{-\frac{1}{2}}\,d(x+4) + c$$

$$= \frac{1}{1-\frac{1}{2}}(x+4)^{-\frac{1}{2}+1} + c \longleftrightarrow = \frac{1}{1-\frac{1}{2}}u^{-\frac{1}{2}+1} + c$$

$$= 2(x+4)^{\frac{1}{2}} + c = 2\sqrt{x+4} + c.$$

To solve $\int (x+4)^{-\frac{1}{2}}dx$ using u-subs.

Let $u = x+4$, then

$$du = dx$$

Then $\int (x+4)^{-\frac{1}{2}}dx$

$$= \int u^{-\frac{1}{2}}du$$

As $y(5)=7$, $y(5) = 2\sqrt{5+4} + c = 7 \Rightarrow c = 1$. Thus $\boxed{y(x) = 2\sqrt{x+4} + 1}$

Exercise 3 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}};\ y(0) = 0$$

ANS: We have

$$y = \int \frac{1}{\sqrt{1-x^2}}\,dx + c$$

$$= \sin^{-1}x + C$$

As $y(0)=0$, we know $y(0) = \underbrace{\cancel{\sin^{-1}0}}_{}{}^{0} + c = 0$, $c = 0$

Thus $y = \sin^{-1}x$