Lecture 1 Differential Equations and Mathematical Models

Differential Equations

Changing Quantities

- The laws of the universe are written in the language of mathematics.
- Algebra is sufficient to solve many static problems,
- but the most interesting natural phenomena involve change and are described by equations that relate to changing quantities.

Number of rabbits Saving account balance Position of an object

$$\text{Rate of change} = \frac{\text{change of } x}{\text{change of } t} = \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Derivative as Rate of Change

- Because the derivative $dx/dt = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t.
- It is natural that equations involving derivatives are frequently used to describe the changing universe.
- What is a **differential equation**?

  An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

Order of a differential eqns is the order of the highest derivative present in the eqn.

Examples of differential equations

- The equation

$$\frac{dx}{dt} = x^2 + t^2 \tag{1}$$

involves the unknown function $x(t)$ and its first derivative $x'(t)$.

- The equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0 \tag{2}$$

(second-order diff. eqn)

involves the unknown function $y(x)$ and its first two derivatives.

Goals of the Study of Differential Equations

Three Goals

The study of differential equations has three principal goals:

1. To **discover** the differential equation that describes a specified physical situation.
2. To **find** - either exactly or approximately - the appropriate solution of that equation.
3. To **interpret** the solution that is found.

Unknowns

- In algebra, we typically seek the unknown numbers that satisfy an equation such as

$$x^3 + 7x^2 - 11x + 41 = 0.$$

- By contrast, in solving a differential equation, we are challenged to find the unknown functions $y = y(x)$ for which an identity such as $y'(x) = 2xy(x)$ - that is, the differential equation

$$\frac{dy}{dx} = 2xy$$

holds on some interval of real numbers.

- Ordinarily, we will want to find *all solutions* of the differential equation, if possible.

Example 1 Substitute $y = e^{rt}$ into the given differential equation to determine all values of the constant r for which $y = e^{rt}$ is a solution of the equation.

$$y'' + 3y' - 4y = 0 \quad (*)$$

ANS: If $y(t) = e^{rt}$, then

$y'(t) = (e^{rt})' = (rt)'e^{rt} = re^{rt}$

$y''(t) = (re^{rt})' = r^2 e^{rt}$

Chain Rule:

$[f(g(t))]' = f'(g(t)) \cdot g'(t)$

Eg: $(\sin(2t))' = \cos(2t) \cdot 2$

Plug $y(t)$, $y'(t)$, $y''(t)$ into $(*)$.

$r^2e^{rt} + 3 \cdot r e^{rt} - 4e^{rt} = 0 \Rightarrow (r^2 + 3r - 4)e^{rt} = 0$

As $e^{rt} \neq 0$, we have $r^2 + 3r - 4 = (r+4)(r-1) = 0$

$\Rightarrow r = -4$ or $r = 1$. Thus $y(t) = e^{-4t}$ and $y(t) = e^{t}$ are solutions to $(*)$

Example 2 Verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial conditon.

$y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$

ANS: $LHS = y'(x) = (Ce^{-x} + x - 1)' = -Ce^{-x} + 1$

$\quad \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$RHS = x - y = x - (Ce^{-x} + x - 1) = -Ce^{-x} + 1$

Thus y satisfies the given diff. eqn.

Since $y(0) = 10$.

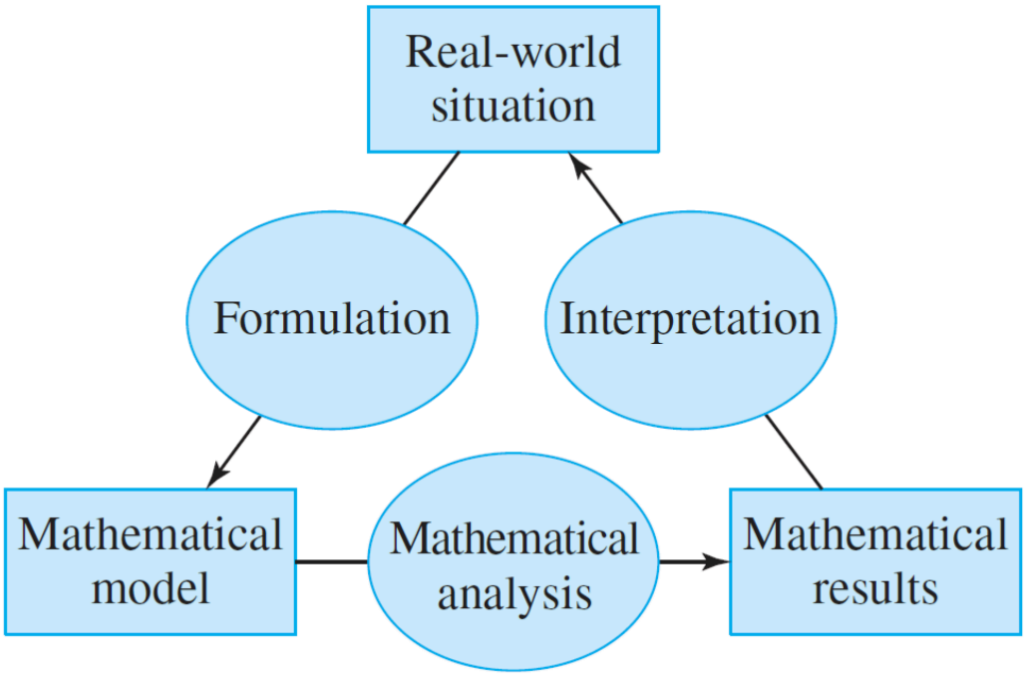
$y(0) = C \cdot e^{-0} + 0 - 1 = C - 1 = 10$

$\Rightarrow C = 11$

Thus we have $y = 11e^{-x} + x - 1$

Mathematical Models

The Process of Mathematical Modeling



- The following example (**Example 3**) illustrates the process of translating scientific laws and principles into differential equations.
- We will see more mathematical models throughout this semester.

$N'(t) = \frac{dN}{dt}$

Example 3 In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not. Set up a differential equation for N.

$N(t)$ $P-N(t)$ multiply by a constant k

ANS

We know

- The # of persons with the diseas: $N(t)$
- The rate of change of $N(t)$: $\frac{dN(t)}{dt} = N'(t)$
- The # who do not have the disease: $P-N(t)$

$$N'(t) = \frac{dN}{dt} = k \cdot N(t) \cdot (P - N(t)) \Rightarrow \frac{dN}{dt} = kN(P-N)$$