From 3-manifolds to modular categories

Qing Zhang

UC Santa Barbara

Motivation

- Tensor categories $\to$ invariants of 3-manifolds
- 3 manifolds $\to$ tensor categories?
- First proposed by Cho-Gang-Kim JHEP 2020, 115(2020)
- Mathematically improved by Cui-Qiu-Wang arXiv: 2101.01674
  - Representations $\rho : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C}) \quad \rightsquigarrow \quad$ simple objects
  - classical Chern-Simons invariant $\quad \rightsquigarrow \quad$ twist
  - adjoint-Reidemeister torsion $\quad \rightsquigarrow \quad$ quantum dim
- "Inverse problem" of Volume Conjecture?

Correspondence based on (S, T)

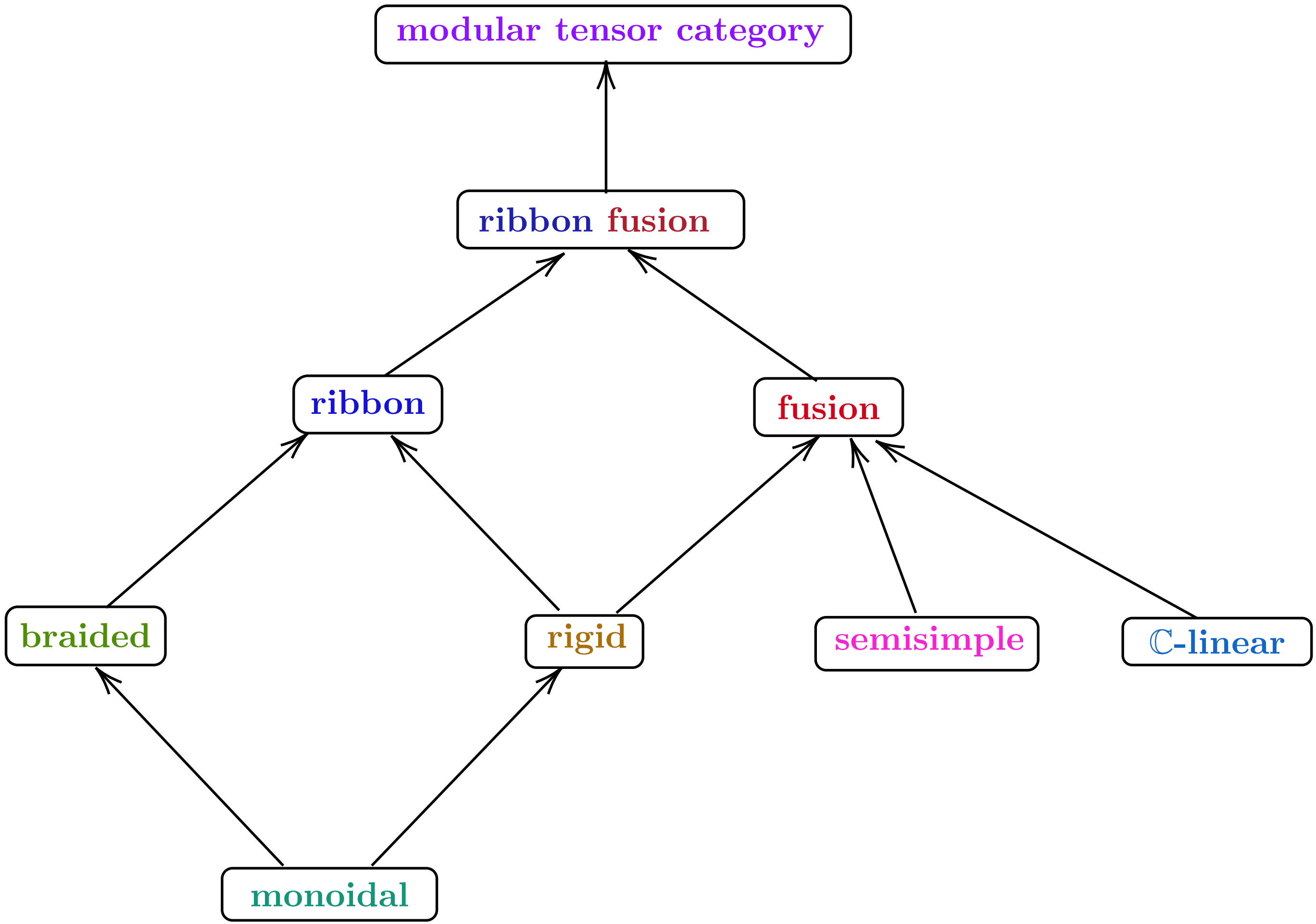
Theorem (Cui-Qiu-Wang)

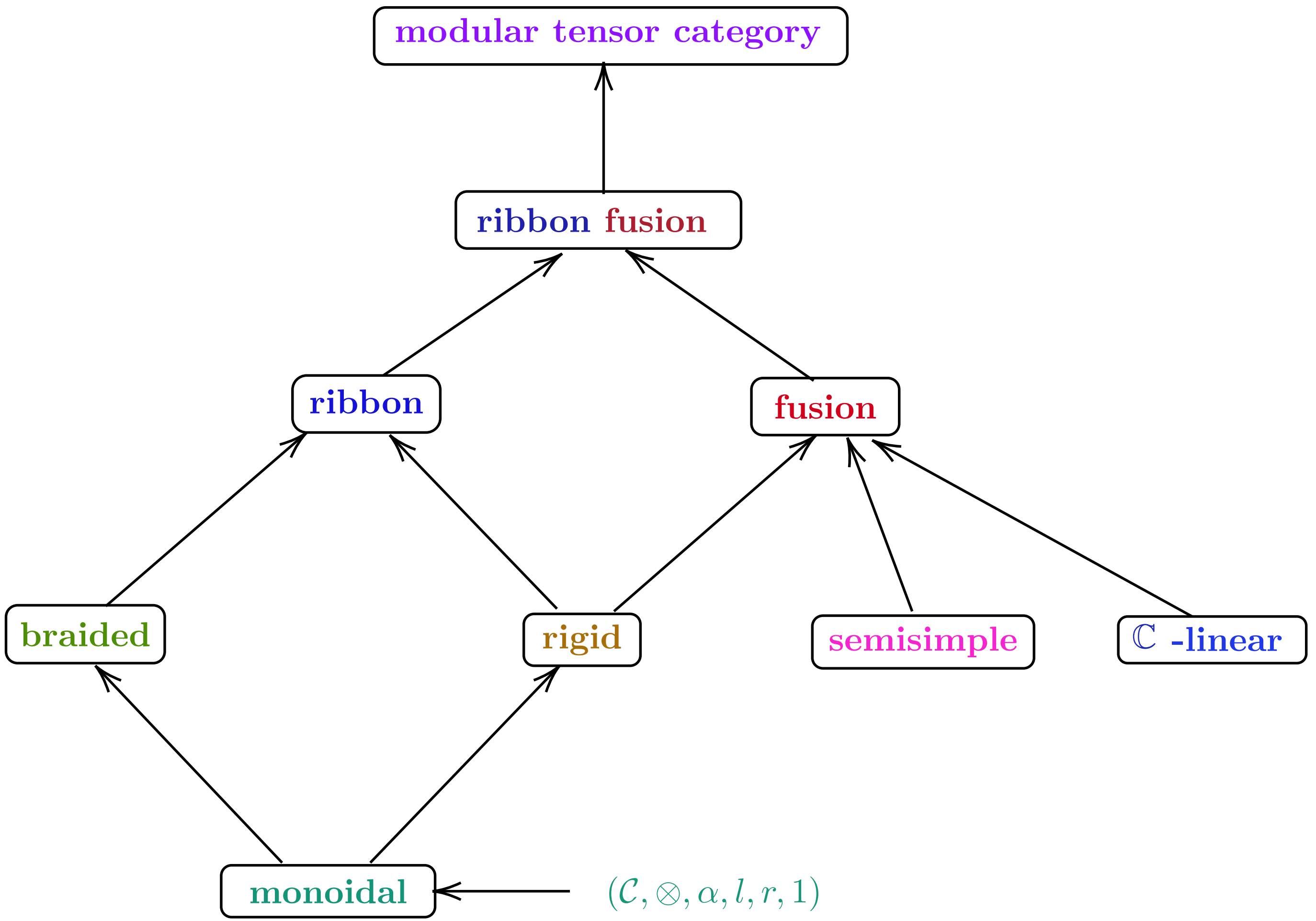
Let M be a Seifert fibered space with three singular fibers.
The ribbon fusion category constructed from M is

$$\mathcal{B}_M := \left(\boxtimes_{k=1}^{3} \mathsf{TLJ}\left(A_k\right)^e\right) \bigoplus \left(\boxtimes_{k=1}^{3} \mathsf{TLJ}\left(A_k\right)^o\right)$$

Theorem (Cui-Gustafson-Qiu-Z)

For a general torus bundle over S^1 with SOL geometry, there exists a finite abelian group G and a quadratic form q such that the corresponding ribbon fusion category is the $\mathbb{Z}_2$-equivariantization of $\mathcal{C}(G, q)$.


modular tensor category
ribbon fusion
ribbon
fusion
braided
rigid
semisimple
ℂ-linear
monoidal



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monoidal
$(\mathcal{C}, \otimes, \alpha, l, r, 1)$

Rep(G)

A monoidal category $(\mathcal{C}, \otimes, \alpha, l, r, \mathbf{1})$, where

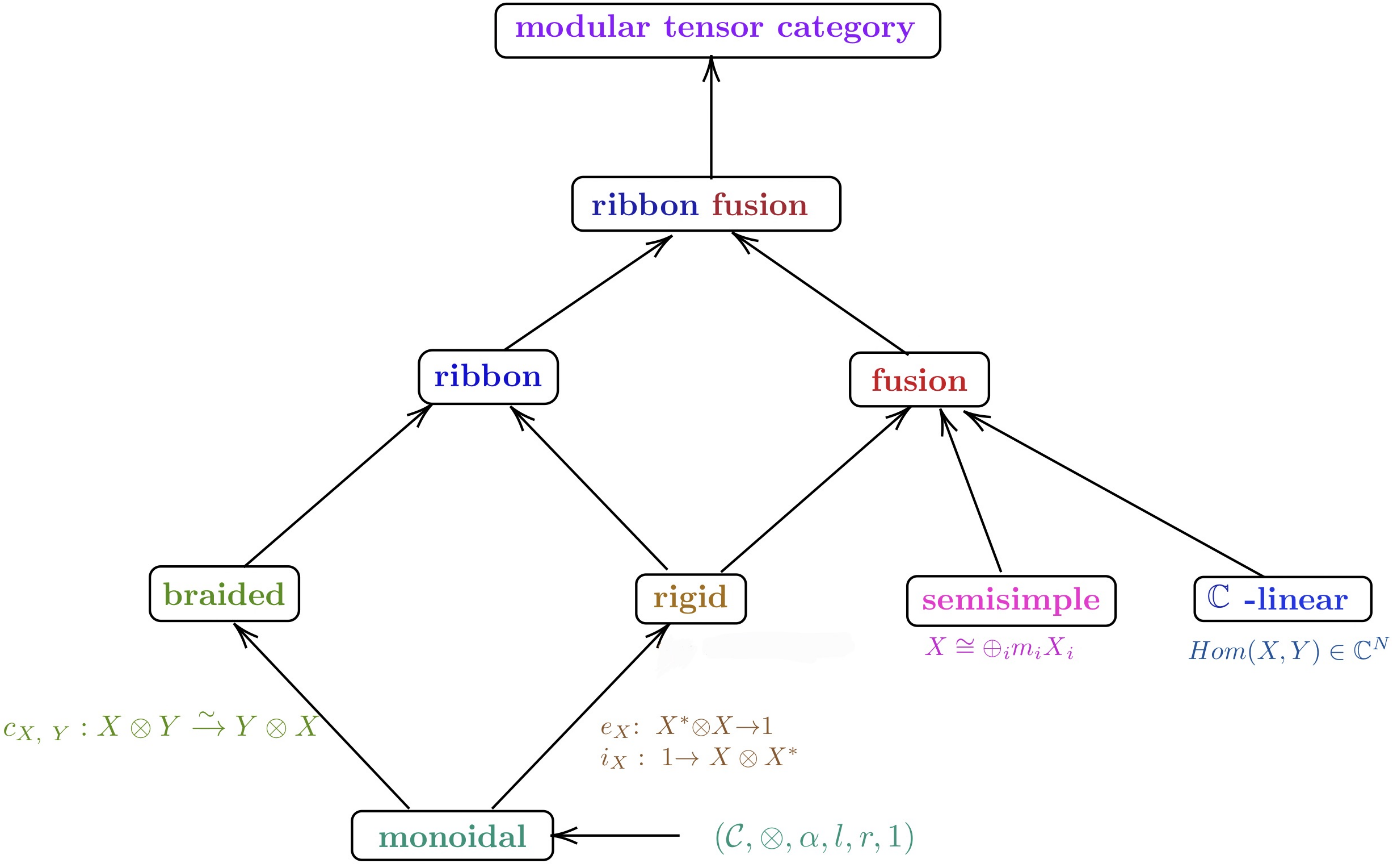
$$l_X : \mathbf{1} \otimes X \xrightarrow{\sim} X \text{ (left unitor)} \qquad r_X : X \otimes \mathbf{1} \xrightarrow{\sim} X \text{ (right unitor)}$$
$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \text{ (associator)}$$

Let G be a finite group.

$\mathcal{C} = \mathrm{Rep}(G)$, category of finite dimensional representations of G over $\mathbb{C}$.

- Monoidal
- $\mathrm{End}(X) \cong \mathbb{C}$ if X is irreducible (Shur's lemma)
- Semisimple: $X \cong \oplus_i m_i X_i$ (Maschke's theorem)
- $\mathbb{C}$-linear: $\mathrm{Hom}(X, Y) = \mathbb{C}^N$.
- Rigid

$$e_X : X^* \otimes X \to \mathbf{1}, \; i_X : \mathbf{1} \to X \otimes X^*$$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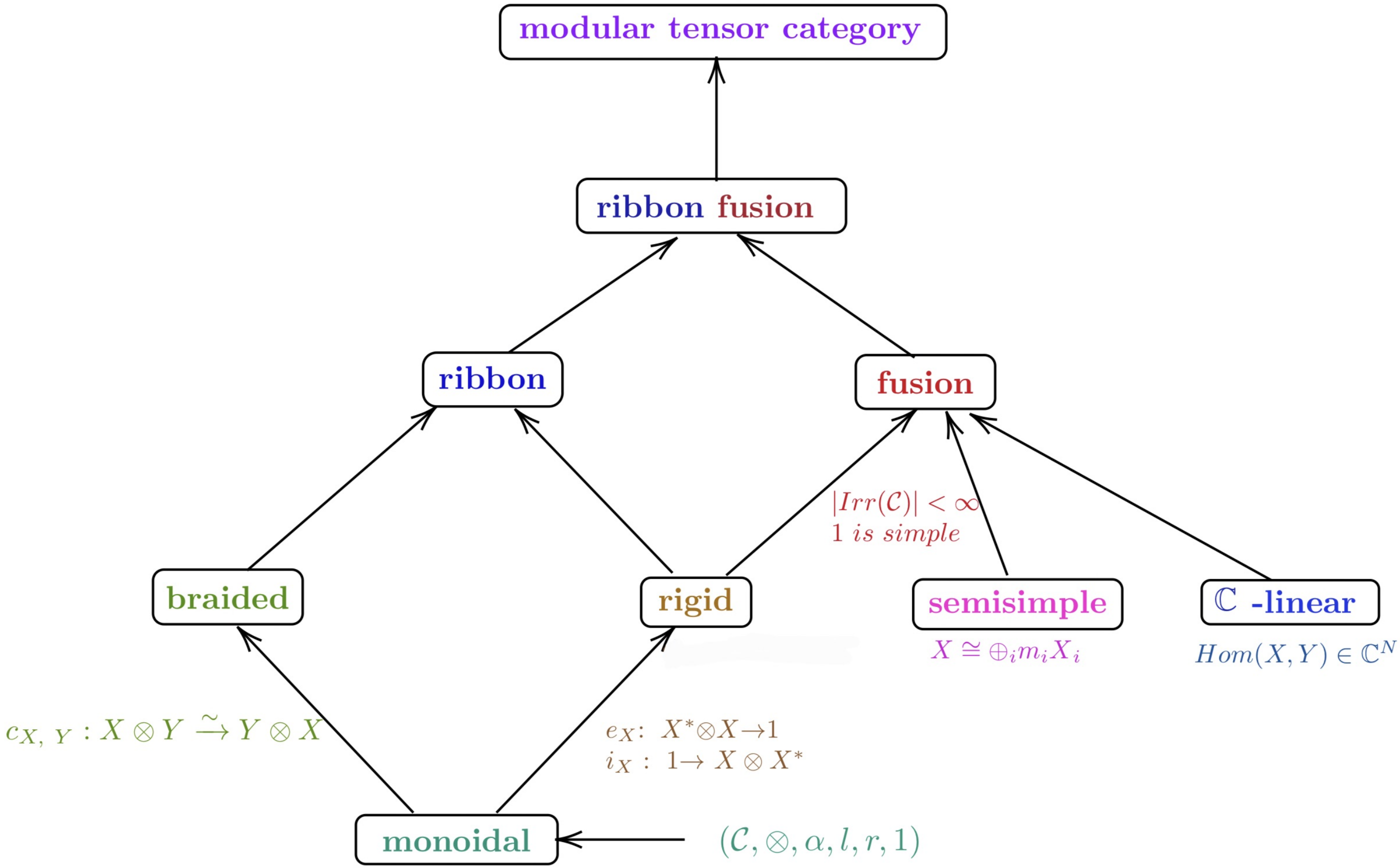
$\mathbb{C}$ -linear
$X \cong \oplus_i m_i X_i$
$Hom(X,Y) \in \mathbb{C}^N$
$c_{X,\,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$
$e_X\colon\ X^* \otimes X \to 1$
$i_X :\ 1 \to X \otimes X^*$
monoidal
$(\mathcal{C}, \otimes, \alpha, l, r, 1)$



modular tensor category
ribbon fusion
ribbon
fusion
$|Irr(\mathcal{C})| < \infty$
1 is simple
braided
rigid
semisimple
$\mathbb{C}$ -linear
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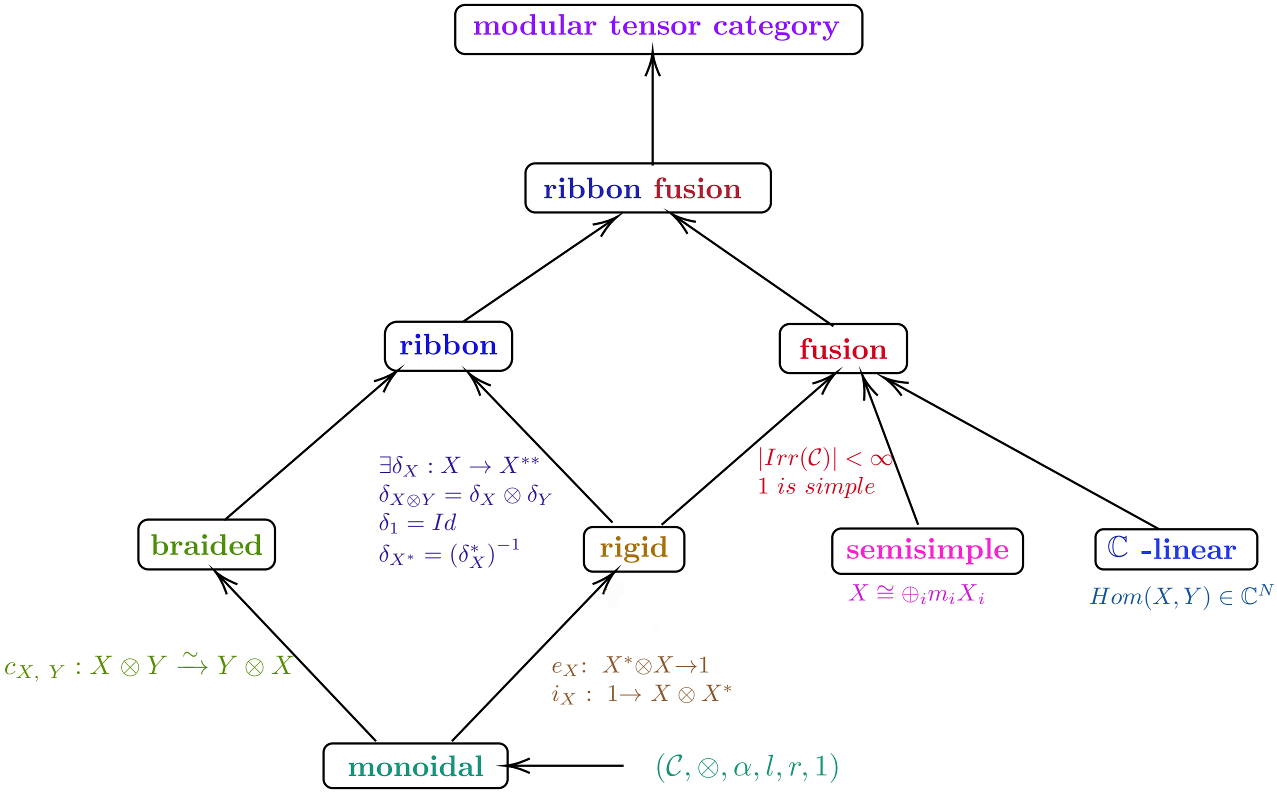
modular tensor category
ribbon fusion
ribbon
fusion
$\exists\delta_X : X \to X^{**}$
$\delta_{X\otimes Y} = \delta_X \otimes \delta_Y$
$\delta_1 = Id$
$\delta_{X^*} = (\delta_X^*)^{-1}$
$|Irr(\mathcal{C})| < \infty$
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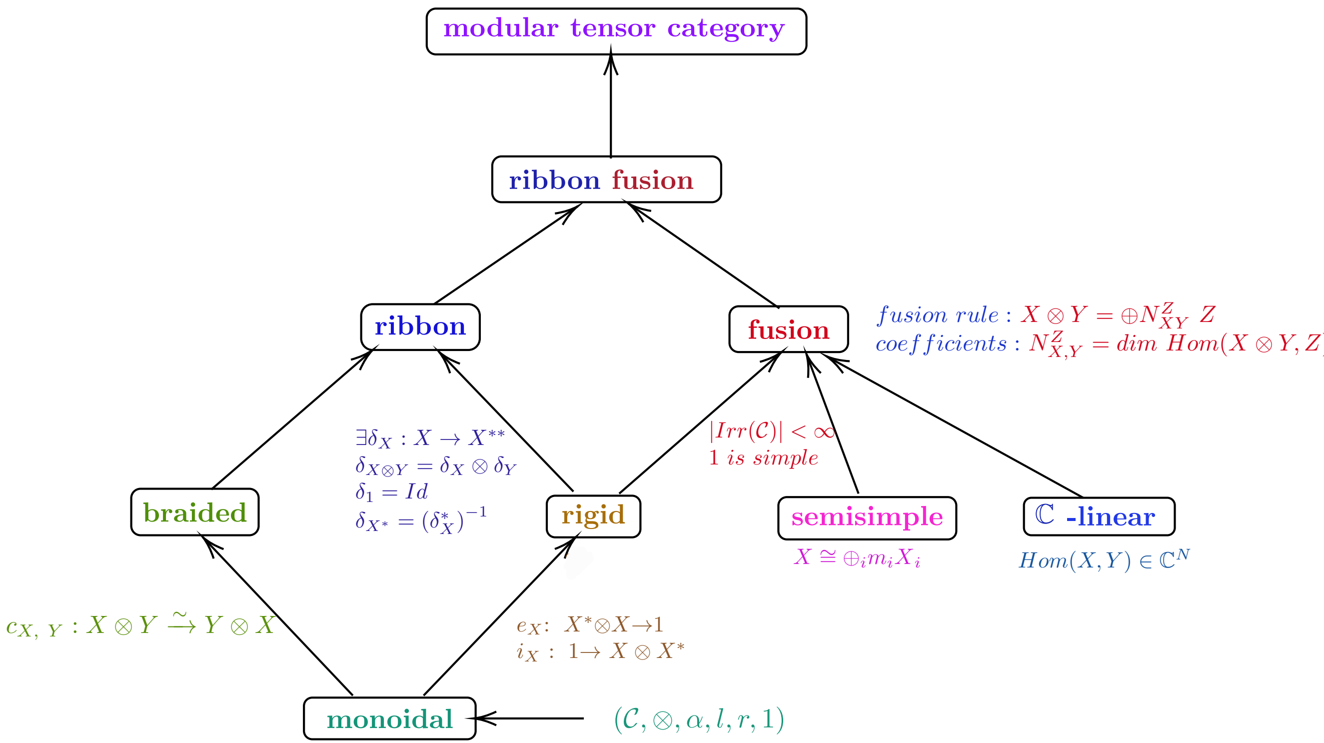
Fusion rules

In a **fusion** category, we have

- *fusion rule* : $X \otimes Y = \oplus N_{XY}^{Z}\ Z$
- *coefficients* : $N_{X,Y}^{Z} = dim\ Hom(X \otimes Y, Z)$



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S and T matrix

In a **ribbon** **fusion** category, let $X_i \in \mathrm{Irr}(\mathcal{C})$.

- T matrix

  $\theta_X = \psi_X \delta_X : X \xrightarrow{\sim} X$, where $\psi_X : X^{**} \to X$.

  $\mathrm{End}(X_i) = \mathbb{C}$

  $\theta_{X_i} = \theta_i \, \mathrm{Id}_{X_i}, \ \theta_i \in \mathbb{C}$

  $T = \mathrm{diag}(\theta_i)$

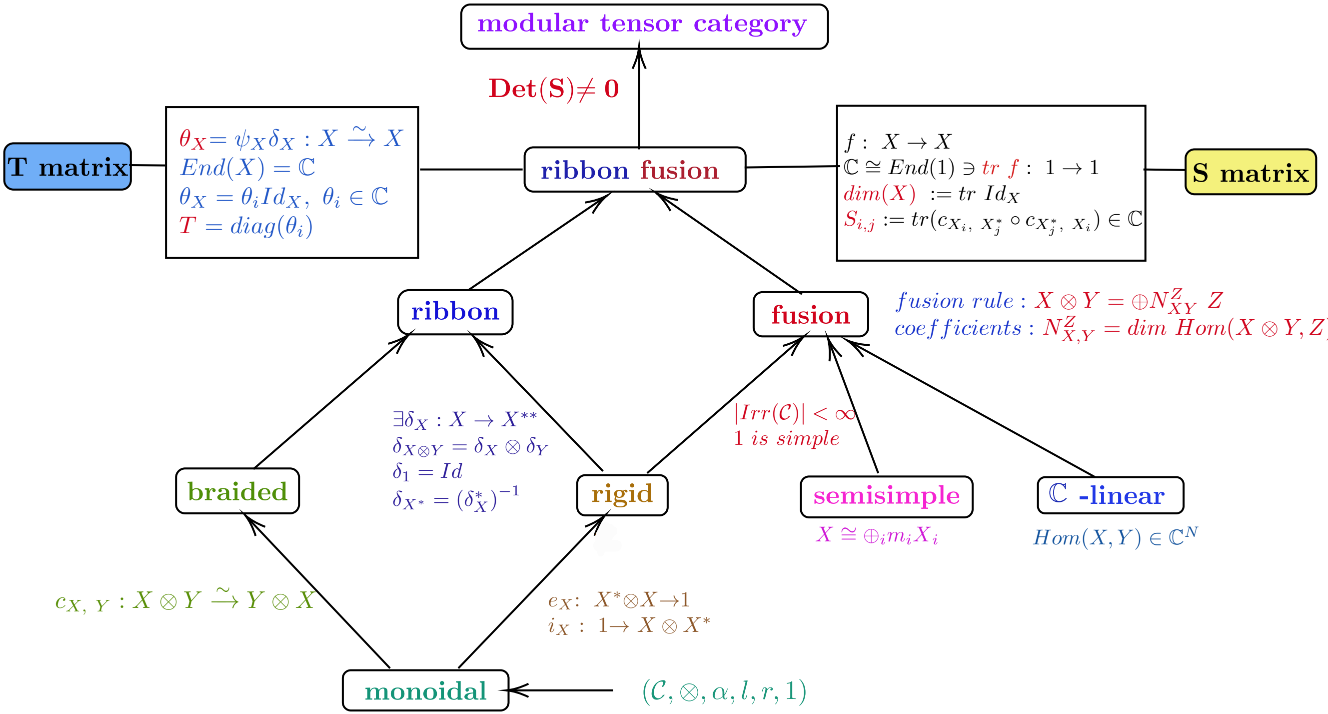
- S matrix

  $f : X \to X$

  $\mathbb{C} \cong End(1) \ni tr\ f : \mathbf{1} \to \mathbf{1}$

  $\dim(X_i) := tr\ Id_{X_i}, \quad D^2 = \sum \dim(X_i)^2$

  $S_{i,j} := tr(c_{X_i, X_j^*} \circ c_{X_j^*, X_i}) \in \mathbb{C}$



modular tensor category
Det(S)≠ 0
T matrix
θ_X= ψ_X δ_X : X ≃→ X
End(X) = ℂ
θ_X = θ_i Id_X, θ_i ∈ ℂ
T = diag(θ_i)
ribbon fusion
f : X → X
ℂ ≅ End(1) ∋ tr f : 1 → 1
dim(X) := tr Id_X
S_{i,j} := tr(c_{X_i, X_j^*} ∘ c_{X_j^*, X_i}) ∈ ℂ
S matrix
ribbon
fusion
fusion rule : X ⊗ Y = ⊕N^Z_{XY} Z
coefficients : N^Z_{X,Y} = dim Hom(X ⊗ Y, Z)
∃δ_X : X → X^{**}
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X ≅ ⊕_i m_i X_i
Hom(X, Y) ∈ ℂ^N
c_{X, Y} : X ⊗ Y ≃→ Y ⊗ X
e_X: X^*⊗X→1
i_X : 1→ X ⊗ X^*
monoidal
(C, ⊗, α, l, r, 1)

Examples of modular categories

- Pointed: $\mathcal{C}(A, Q)$, finite abelian group A, non-deg. quadratic form Q on A.
- The Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$.

  Objects of $\mathcal{Z}(\mathcal{C})$ are (Z, γ), where Z is an object of $\mathcal{C}$ and γ is half braiding.
- From quantum groups:

$$\mathfrak{g} \rightsquigarrow U_q\mathfrak{g} \overset{q = e^{\pi i/l}}{\rightsquigarrow} \mathrm{Rep}(U_q\mathfrak{g}) \overset{/\langle Ann(Tr)\rangle}{\rightsquigarrow} \mathcal{C}(\mathfrak{g}, l)$$

$\mathrm{SL}_2(\mathbb{Z})$ representation

Given a modular category with (unnormalized) modular data (S, T).

- $S^4 = \dim(\mathcal{C})^2\,\mathrm{Id}$, $(ST)^3 = p^+\,S^2$, where $p^\pm = \sum_i \theta_i^\pm d_i^2$.
- $\mathrm{SL}_2(\mathbb{Z}) = \langle \mathfrak{s}, \mathfrak{t} \rangle$, where $\mathfrak{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathfrak{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- $\mathfrak{s}^4 = \mathrm{Id}$, $(\mathfrak{s}\mathfrak{t})^3 = \mathfrak{s}^2$.
- $\mathfrak{s} \mapsto S$, $\mathfrak{t} \mapsto T$ gives a projective representation $\bar{\rho}_{\mathcal{C}}$ of $\mathrm{SL}_2(\mathbb{Z})$.
- [Ng-Schauenburg '10]

  If $N = \mathrm{ord}(T)$,
  - $\bar{\rho}_{\mathcal{C}}$ factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.
  - $\mathbb{Q}(S) \subset \mathbb{Q}(\zeta_N)$, $N = \mathrm{ord}(T)$.

$\mathrm{SL}_2(\mathbb{Z})$ representation (continued)

- $s := \dfrac{1}{\sqrt{\dim(\mathcal{C})}} S$ and $t := \dfrac{1}{\gamma} T$, where γ is any third root of the multiplicative central charge $\xi = p^+(\mathcal{C})/\sqrt{\dim(\mathcal{C})}$.
- $\mathfrak{s} \mapsto s$, $\mathfrak{t} \mapsto t$ gives a linear representation ρ of $\mathrm{SL}_2(\mathbb{Z})$.
- [Dong-Lin-Ng '15]

  If $n = \mathrm{ord}(t)$,

  - ρ factors through $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$
  - $\mathrm{im}(\rho) \subset \mathrm{GL}_r(\mathbb{Q}(\zeta_n))$

$\mathrm{SL}_2(\mathbb{Z})$ representation (continued)

- ρ factors through $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.
- Chinese Reminder Theorem $\to \mathrm{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$.
- [Nobs 1976, Nobs and Wolfart 1976] Irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$ are classified using subrepresentations of Weil representations.
- [Ng-Rowell-Wang-Wen '22] Reconstruction of modular data from irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

  Classification up to modular data, rank $= 6$.

Chern-Simons invariant

- X: a closed oriented 3-manifold
- $\rho : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$
- $\chi_\rho = \mathrm{Tr}_\rho : \pi_1(X) \to \mathbb{C}$
- Chern-Simons invariant
$$\mathrm{CS}(\rho) = \frac{1}{8\pi^2} \int_X \mathrm{Tr}(dA_\rho \wedge A_\rho + \frac{2}{3} A_\rho \wedge A_\rho \wedge A_\rho) \mod 1,$$
where $A_\rho \in \Omega^1(X; \mathfrak{sl}_2)$ connection 1-form with holonomy ρ.

Adjoint Reidemeister Torsion

- $C = (0 \longrightarrow C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0)$ is acyclic if $H_i(C) = 0$.
- Fix a basis c_i of C_i.

  Choose a basis b_i of $\mathrm{Im}(\partial_i)$, $b_i \sqcup \tilde{b}_{i-1}$ is a basis of C_i.
- Let D_i be the transition matrix from $b_i \sqcup \tilde{b}_{i-1}$ to c_i.
- Torsion of C

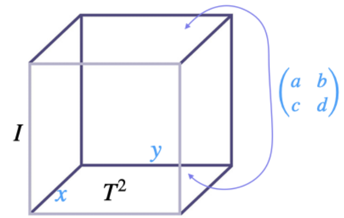
$$\tau(C, c) := \left| \prod_{i=0}^{n} \det\left(D_i\right)^{(-1)^{i+1}} \right|$$

- $\rho : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$
- $C(\tilde{X}) \otimes_{\mathrm{adj}_\rho} \mathbb{C}^3 = \left(\cdots \to C_i(\tilde{X}) \otimes_{\mathrm{adj}_\rho} \mathbb{C}^3 \xrightarrow{\partial_i} C_{i-1}(\tilde{X}) \otimes_{\mathrm{adj}_\rho} \mathbb{C}^3 \to \cdots \right)$
- Adjoint Reidemeister Torsion

$$\mathrm{Tor}(\rho) := \tau\left(C(\tilde{X}) \otimes_{\mathrm{adj}_\rho} \mathbb{C}^3 \right)$$

Torus bundles over the circle

- Let M be a torus bundle over S^1 with the monodromy map



$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, where $|a + d + 2| > 4$.

- $N := |a + d + 2| > 4 \quad \Leftrightarrow \quad$ SOL geometry.
- Its fundamental group has the presentation, $\pi_1(M) = \langle x, y, h \mid x^a y^c = h^{-1}xh, x^b y^d = h^{-1}yh, xyx^{-1}y^{-1} = 1 \rangle$
- An example:

  Consider M to be a torus bundle over S^1 with $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

  We will use this example as we go through the program.

Program: Simple object types

- $\mathcal{R}(X) = \{\rho : \pi_1(X) \mapsto \mathrm{SL}(2,\mathbb{C})\}$ representation variety
- $\chi(X) = \{\chi_\rho \mid \rho \in \mathcal{R}(X)\}$ character variety. Call ρ a preimage of χ_ρ

Definition

- A $\chi \in \chi(X)$ is non-Abelian, if at least one preimage ρ is non-Abelian, i.e., it has non-Abelian image in $\mathrm{SL}(2,\mathbb{C})$.
- A non-Abelian χ is adjoint-acyclic, if all of its non-Abelian preimages ρ are adjoint-acyclic.

- Postulate 1: a simple object type is an adjoint-acyclic non-Abelian χ.
- Postulate 2: a label set $L(X)$ is a finite set of simple object types with a prechosen type χ_0 such that

$$CS(\chi) - CS(\chi_0) \in \mathbb{Q}, \quad \forall \chi \in L(X).$$

The tensor unit is χ_0.

Program: Simple object types (continued)

Example (Label set of M)

- $\pi_1(M) = \langle x, y, h \mid x^2 y h^{-1} x h, xy = h^{-1} y h, xy = yx \rangle$
- Irreducibles $\rho_k, \quad k = 1, 2$
  $$x \mapsto \begin{pmatrix} e^{\frac{2\pi ik}{5}} & 0 \\ 0 & e^{-\frac{2\pi ik}{5}} \end{pmatrix}, y \mapsto \begin{pmatrix} e^{\frac{4\pi ik}{5}} & 0 \\ 0 & e^{-\frac{4\pi ik}{5}} \end{pmatrix}, h \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
- Reducibles $\rho_\pm$
  $$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, h \mapsto \begin{pmatrix} v_\pm & 0 \\ 0 & v_\pm^{-1} \end{pmatrix},$$
  with $u = \frac{\sqrt{5}-1}{2}, \quad v_\pm = \pm\frac{\sqrt{5}-1}{2}$.
- $L(M) = \{\rho_+, \rho_-, \rho_1, \rho_2\} \quad \rho_0 := \rho_+$ Identifying ρ with χ_ρ
- $\mathrm{CS} = \left\{0, 0, \frac{1}{5}, -\frac{1}{5}\right\} \qquad \mathrm{Tor} = \left\{5, 5, \frac{5}{4}, \frac{5}{4}\right\}$

Program: Simple object types (continued)

$$H^1(X;\mathbb{Z}_2) = \mathrm{Hom}(\pi_1(X),\mathbb{Z}_2)$$
$$= \{\sigma : \pi_1(M) \to \{\pm I_2\} \subset \mathrm{SL}(2,\mathbb{C})\} \quad \text{central representations}$$

- $H^1(X;\mathbb{Z}_2) \curvearrowright \mathcal{R}(X) \quad (\sigma.\rho)(\cdot) := \sigma(\cdot)\rho(\cdot) \quad \Rightarrow \quad H^1(X;\mathbb{Z}_2) \curvearrowright \chi(X)$
- $s(X) := H^1(X;\mathbb{Z}_2)\,\rho_0 \subset L(X)$
- The label set needs to satisfy:

$$\sum_{\rho_\alpha \in L(X)} \frac{1}{2\,\mathrm{Tor}(\rho_\alpha)} = 1, \qquad \left| \sum_{\rho_\alpha \in L(X)} \frac{\exp(-2\pi i\,\mathrm{CS}(\rho_\alpha))}{2\,\mathrm{Tor}(\rho_\alpha)} \right| = \frac{\sqrt{|s(X)|}}{\sqrt{2\,\mathrm{Tor}(\rho_0)}}$$

Example (Label set of M (continued))

The two conditions are satisfied

$$H^1(M;\mathbb{Z}_2) = \mathbb{Z}_2 \qquad \rho_+ \leftrightarrow \rho_- \qquad s(X) = \{\rho_+, \rho_-\}$$

Program: Twists and dimensions

Postulate 3: the twist is

$$\theta_\alpha = e^{-2\pi i(\mathrm{CS}(\rho_\alpha)-\mathrm{CS}(\rho_0))}$$

Postulate 4: the quantum dimension is

$$D^2 = 2\,\mathrm{Tor}\,(\rho_0) \qquad \frac{d_\alpha^2}{D^2} = \frac{1}{2\,\mathrm{Tor}\,(\rho_\alpha)}$$

Example (Twists and dimensions for M)

- $L(M) = \{\rho_+, \rho_-, \rho_1, \rho_2\} \quad \rho_0 := \rho_+$
- $\mathrm{CS} = \left\{0, 0, \frac{1}{5}, -\frac{1}{5}\right\} \quad \mathrm{Tor} = \left\{5, 5, \frac{5}{4}, \frac{5}{4}\right\}$
- $\theta = \left\{1, 1, e^{-\frac{2\pi i}{5}}, e^{\frac{2\pi i}{5}}\right\}$
- $D = \sqrt{10} \quad |d| = \{1, 1, 2, 2\}$

Program: S-matrix

Definition

- A loop operator is a pair (a, R) where a is a conjugacy class of $\pi_1(X)$, and R is an irrep of $\mathrm{SL}(2, \mathbb{C})$.
- The weight of $\rho \in \mathcal{R}(X)$ w.r.t (a, R) is

$$W_\rho(a, R) := \mathrm{Tr}_R(\rho(a))$$

For example,

$$\mathrm{Sym}^j := (j+1)\text{-dim irrep}, \qquad W_\rho\left(a, \mathrm{Sym}^1\right) = \mathrm{Tr}(\rho(a)) = \chi_\rho(a)$$

- Postulate 5: each type is associated with some loop operators,

$$\rho_\alpha \mapsto \{(a_\alpha^\kappa, R_\alpha^\kappa)\}_\kappa \qquad \text{guess-and-trial}$$

$$W_\beta(\alpha) := \prod_\kappa W_{\rho_\beta}\left(a_\alpha^\kappa, R_\alpha^\kappa\right) = \prod_\kappa \mathrm{Tr}_{R_\alpha^\kappa}\left(\rho_\beta\left(a_\alpha^\kappa\right)\right), \quad \rho_\alpha, \rho_\beta \in L(X)$$

Program: S-matrix (continued)

Postulate 6: the un-normalized S-matrix is given by,

$$\tilde{S}_{\alpha\beta} = W_\beta(\alpha) W_0(\beta) \quad \Leftrightarrow \quad W_\beta(\alpha) = \frac{\tilde{S}_{\alpha\beta}}{\tilde{S}_{0\beta}}$$

In particular, $d_\alpha = W_0(\alpha)$.

Example (S-matrix for M)

$$\rho_\pm \mapsto \left(x, \mathrm{Sym}^0\right), \qquad \rho_k \mapsto \left(x^{-3k}, \mathrm{Sym}^1\right), k = 1, 2$$

$$W_0(k) = \mathrm{Tr}_{\mathrm{Sym}^1}\left(\rho_0\left(x^{-3k}\right)\right) = 2$$

$$W_j(k) = \mathrm{Tr}_{\mathrm{Sym}^1}\left(\rho_j\left(x^{-3k}\right)\right) = 2\cos\frac{4\pi jk}{5}$$

$$\tilde{S} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 4\cos\frac{4\pi}{5} & 4\cos\frac{2\pi}{5} \\ 2 & 2 & 4\cos\frac{2\pi}{5} & 4\cos\frac{4\pi}{5} \end{pmatrix}, \quad T = \mathrm{diag}(1, 1, e^{-\frac{2\pi i}{5}}, e^{\frac{2\pi i}{5}}),$$

which corresponds to a subcategory of $\mathcal{C}(\mathfrak{so}_5, e^{-\frac{3\pi i}{10}}, 10)$.

Modular data from torus bundle over S^1

Theorem (Cui-Qiu-Wang)

Let M be the torus bundle over S^1, with the monodromy matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $N = a + d + 2 > 4$ is odd and (c, N) are coprime. The modular data constructed from M matches that of $\mathcal{C}(\mathfrak{so}_N, q, 2N)_{ad}$.

Theorem (Cui-Gustafson-Qiu-Z)

For a general torus bundle over S^1 with SOL geometry, there exists a finite abelian group G and a quadratic form q such that the corresponding ribbon fusion category is the $\mathbb{Z}_2$-equivariantization of $\mathcal{C}(G, q)$.

(S, T) from Seifert fibered space

Theorem (Cui-Qiu-Wang)

Let M be a Seifert fibered space with three singular fibers.
The ribbon fusion category constructed from M is

$$\mathcal{B}_M := \left(\boxtimes_{k=1}^{3} \mathrm{TLJ}\left(A_k\right)^e\right) \bigoplus \left(\boxtimes_{k=1}^{3} \mathrm{TLJ}\left(A_k\right)^o\right)$$

- Conjecturally, the resulting category from a general M is modular iff $H^1\left(M; \mathbb{Z}_2\right) = 0$.

Future questions

- For hyperbolic 3-manifolds?

Conjecture: (W.-Yang '21)

Suppose for r sufficiently large, a hyperbolic cone metric on M with singular locus L and cone angles $\theta^{(r)}$ exists. We denote M with such a hyperbolic cone metric by $M^{(r)}$.

As r varies over all positive odd integers, the relative Reshetikhin-Turaev invariants

$$\mathrm{RT}_r(M, L, \mathbf{m}^{(r)}) = C \frac{e^{\frac{1}{2}\sum_{k=1}^{n}\mu_k \mathrm{H}^{(r)}(\gamma_k)}}{\sqrt{\pm\mathbb{T}_{(M\setminus L,\nu)}([\rho_{M^{(r)}}])}} e^{\frac{r}{4\pi}\left(\mathrm{Vol}(M^{(r)})+i\mathrm{CS}(M^{(r)})\right)}\left(1+O\left(\frac{1}{r}\right)\right),$$

where C is a quantity of norm 1 independent of the geometric structure on M.

- Construction beyond modular data. E.g., F-matrices, R-matrices.
- Operations on MTCs $\leftrightarrow$ Constructions of 3-manifolds.
- ......

Thank you!