Review of Common Ordinary Differential Equations

Additional Notes Summarized by Yourself

You can fill in this empty block to summarize the course contents that are not listed in this file.

Separable Equations

Eg: Spring 2019 #3

Fall 2019 #3

Spring 2015 #2

Separable Equations: $\frac{dy}{dx} = g(x)k(y)$

Solution: $\int \frac{dy}{k(y)} = \int g(x)dx + C$

Also check if $k(y) = 0$ is a solution

Applications: Newton's law of cooling: $\frac{dT}{dt} = k(A - T)$

Logistic equations: $\frac{dP}{dt} = kP(M - P) = aP - bP^2$

Linear First-order Equations

Eg: Spring 2019 #2, Spring 2018 #6

Fall 2018 #6

Fall 2015 #1

Linear First-order Equations: $\frac{dy}{dx} + P(x)y = Q(x)$

Solution: $\rho y = \int \rho Q(x)dx$, where $\rho = e^{\int P(x)dx}$.

Applications: Mixture Problems: $\frac{dx}{dt} = r_i c_i - r_o c_o$,

Fall 2018 #2

Fall 2015 #4

where $c_o(t) = \frac{x(t)}{V(t)}$, $V(t) = V_0 + (r_i - r_0)\, t$

Exact Equations

Eg. Spring 2019 #4 Spring 2018 #7 Fall 2019 #5

Exact Equations: $M(x,y)dx + N(x,y)dy = 0$, where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Solution: $F(x,y) = C$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

Homogeneous Equations

Spring 2018 #5

Fall 2018 #4

Homogeneous Equations: $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

To identify: All $x^n y^m$ have total power $(n + m)$ the same (after rewriting).

Solution: Substitute $v = \frac{y}{x}$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(This converts equation to a separable Diff. E.)

Bernoulli Equations

Fall 2019 #6

Bernoulli Equations: $\frac{dy}{dx} + P(x)y = Q(x)y^n$

Rewrite: $y^{-n}y' + P(x)y^{1-n} = Q(x)$

Solution: $y^{1-n} = v$ and $y^{-n}y' = \frac{1}{1-n}v'$
(This converts equation to a linear Diff. E.)

Reducible Second-order Equations

Reducible Second-order Equations: $F(x, y, y'y'') = 0$

Case 1. y missing: Substitute: $p = y' = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx}$.

Case 2. x missing: Substitute: $p = y' = \frac{dy}{dx}, \quad y'' = p\frac{dp}{dy}$.

Population Models

This topic was covered in Section 2.1. We talked about

- Solving the Logistic Equations.
- How solution curves behave near the equilibrium solutions

See illustrative examples from Lecture Notes Section 2.1.

Acceleration-Velocity Models

This topic was covered in Section 2.3. See the lecture notes and homework questions for examples.

Autonomous Equations and Equilibrium Solutions

Spring 2018 #8
Fall 2019 #2
Fall 2018 #7
Fall 2017 #1
Spring 2015 #2
Spring 2017 #7

Autonomous Equations: $\frac{dx}{dt} = f(x)$

Critical points: values of x such that $f(x) = 0$.
$f(x_0) = 0 \Rightarrow$ equilibrium solution at $x = x_0$
$f(x_0) < 0 \Rightarrow$ solutions go down at $x = x_0$
$f(x_0) > 0 \Rightarrow$ solutions go up at $x = x_0$

Stability of Critical Points: Phase diagram method
unstable = solutions go away (either side)
stable = solutions go towards (both sides)
semi-stable = solutions mixed

Euler's Method

Fall 2019 #7, Fall 2018 #5

Euler's Method: Consider $\frac{dy}{dx} = f(x, y), \quad f(x_0) = y_0$

Euler's method with step size h: $\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h \cdot f(x_n, y_n) \end{cases}$

Existence and Uniqueness Theorem

Eg: Spring 2019 #1

First Order, General Initial Value Problem:

$$y' = f(x, y), \quad y(x_0) = y_0$$

- Solution exists and is unique if f and $\frac{\partial}{\partial y} f$ are continuous at (x_0, y_0).
- Solutions are defined somewhere inside the region containing (x_0, y_0), where f and $\frac{\partial}{\partial y} f$ are continuous.

Linearly Independent Functions

$f_1, \cdots, f_n$ are linearly independent if $c_1 f_1 + \cdots + c_n f_n = 0$ holds if and only if $c_1 = c_2 = \cdots = c_n = 0$.

$$\text{Wronskian: } W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

The Wronskian of n linearly dependent functions $f_1, \cdots, f_n$ is **identically zero**.

2nd Order, Homogeneous Linear, Constant Coefficients

Spring 2018 #10
Fall 2018 #8

2nd Order, Homogeneous Linear, Constant Coefficients:

$$ay'' + by' + cy = 0$$

Characteristic Equation: $ar^2 + br + c = 0$

Solution depends on the type of roots:

- $r = r_1, r_2$(real, not repeated),
  $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
- $r = r_1 = r_1$ (repeated root),
  $y = (c_1 + c_2 x)e^{r_1 x}$.
- $r = r_{1,2} = A \pm Bi$(complex conjugates),
  $y = e^{Ax}(c_1 \cos Bx + c_2 \sin Bx)$

Higher Order, Homogeneous Linear, Constant Coefficients

Higher Order, Homogeneous Linear, Constant Coefficients: $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$

Characteristic Equation: $a_n r^n \cdots + a_1 r + a_0 = 0$

Eg. Spring 2019 #8, Spring 2018 #13

- Solution generalized from 2nd order case.
- Long division method can be used when solving char. eqn. Fall 2017 #12

Reduction of Order

Eg. Spring 2019 #6 Spring 2018 #12

Consider

$$y'' + p(x)y' + q(x)y = 0,$$

with one solution $y = y_1(x)$ known.

Substitute:

$$y = v y_1$$
$$y' = \cancel{v y_1'} + v' y_1$$
$$y'' = \cancel{v y_1''} + 2v' y_1' + v'' y_1$$

Diff. E. becomes $(2v' y_1' + v'' y_1) + p v' y_1 = 0,$

which is separable: $\frac{1}{(v')}(v')' = -\left(p + \frac{2y_1'}{y_1}\right).$

Applications: Euler Equation: $ax^2 y'' + bxy' + cy = 0$

Differential Equations as Vibrations

$$mx'' + cx' + kx = F(t) \begin{cases} m & \text{mass} \\ c & \text{dampening} \\ k & \text{spring constant} \\ F(t) & \text{forcing function} \end{cases}$$

- Free Undamped Motion ($c = 0$ and $F(t) = 0$)
  - General solution $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$, where $\omega_0 = \sqrt{\frac{k}{m}}$.
  - Need to know how to write $x(t) = C\cos(\omega_0 t - \alpha)$, where $C = \sqrt{A^2 + B^2}$ is the amplitude and α is the phase angle. Fall 2015 #13
- Free Damped Motion ($c > 0$ and $F(t) = 0$)

  Fall 2017 #10
  Spring 2015 #13
  - Overdamped (two distinct real roots)
  - Critically damped (repeated real roots)
  - Underdamped (two complex roots)
    The solution can be written as $x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$
- Undamped Forced Oscillations ($c = 0$ and $F(t) \neq 0$)

$$mx'' + kx = F_0 \cos\omega t$$

- Damped Forced Oscillations ($c > 0$ and $F(t) \neq 0$)
  - transient solution $x_{\text{tr}}(t) = x_c(t)$, $x_c(t) \to 0$ as $t \to \infty$
  - steady periodic solution $x_{\text{sp}}(t) = x_p(t)$
  - practical resonance: Consider

$$mx'' + cx' + kx = F_0 \cos\omega t$$

Practical resonance is the maximum value of $C(\omega)$. This may not exist.

Solutions to Nonhomogeneous Equations

Eg. Spring 2019 #9

Consider the nonhomogeneous equation
$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$
with homogeneous solution $y_c = c_1 y_1(x) + \cdots + c_n y_n$ known.
Then the general solution is $y = y_c + y_p$, where y_p is a particular solution.

Undetermined Coefficients: Spring 2018 #9 Fall 2019 #10, Fall 2018 #12
The general nonhomogeneous n th-order linear equation with constant coefficients

Fall 2017 #12

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

Find y_p by guessing a form and then plugging into DE (x^s is chosen so that y_i's are not terms of y_c)

$f(x)$	y_p
$P_m = b_0 + b_2 x + \cdots + b_m x^m$	$x^s\left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m\right)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s\left(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m\right)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\cos kx$ $+(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)\sin kx]$

Variation of Parameters:

$$y'' + P(x)y' + Q(x)y = f(x)$$

homogeneous solution $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$ known.

Eg. Spring 2019 #7
Spring 2018#14

Then a particular solution is

Fall 2019 #11

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

Fall 2018 #9
Fall 2015 #12

Wronskian: $W(x) = y_1 y_2' - y_2 y_1'$.

Remark: Let $u_1 = -\int \frac{y_2(x)f(x)}{W(x)}dx$ and $u_2 = \int \frac{y_1(x)f(x)}{W(x)}dx$, then the above equation becomes

$$y_p(x) = u_1 y_1 + u_2 y_2$$

The Method of Elimination

Examples: $\begin{cases} x' = -3x - 4y \\ y' = 2x + y \end{cases}$, and other examples in Lecture Notes in 4.2.

$\begin{cases} x' = -2y \\ y' = \frac{1}{2}x \end{cases}$, show solutions are ellipses. See Notes in 4.1. Fall 2015 #17

$\begin{cases} x' = y \\ y' = 2x \end{cases}$, show solutions are hyperbolas. See Notes in 4.1.

Constant Coeff. Homogeneous System:

Constant Coeff. Homogeneous: $\dfrac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ Fall 2019 #14

Solution: $\vec{\mathbf{x}} = c_1\vec{\mathbf{x}}_1 + c_2\vec{\mathbf{x}}_2 + \cdots$,
where $\vec{\mathbf{x}}_i$ are fundamental solutions
from eigenvalues & eigenvectors.
The method is described as below.

The Eigenvalue Method for Homogeneous Systems:

The number λ is called an *eigenvalue* of the matrix $\mathbf{A}$ if $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

An *eigenvector* associated with the eigenvalue λ is a nonzero vector $\mathbf{v}$ such that $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

We consider $\mathbf{A}$ to be 2×2, then the general solution is $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{x}}_1(t) + c_2\vec{\mathbf{x}}_2(t)$, with the fundamental solutions $\vec{\mathbf{x}}_1(t), \vec{\mathbf{x}}_2(t)$ found has follows.

Spring 2019 # 17

- Distinct Real Eigenvalues. $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda_1 t}, \vec{\mathbf{x}}_2(t) = \vec{\mathbf{v}}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to remember the method*)

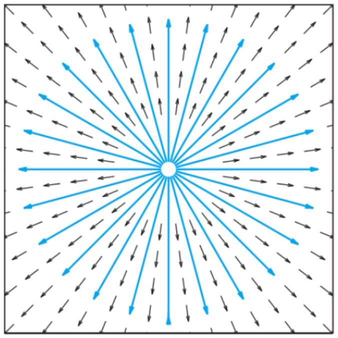
  If $\vec{v} = \vec{a} + i\vec{b}$ is an eigenvector associated with $\lambda = p + qi$, then Spring 2019 #20

  $\vec{\mathbf{x}}_1(t) = e^{pt}\left(\vec{a}\cos qt - \vec{b}\sin qt\right)$, $\vec{\mathbf{x}}_2(t) = e^{pt}(\vec{b}\cos qt + \vec{a}\sin qt)$ Fall 2018 #18
- Defective Eigenvalue with multiplicity 2.
  Find nonzero $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ such that $(\mathbf{A} - \lambda\mathbf{I})^2\vec{\mathbf{v}}_2 = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$.
  Then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda t}$, $\vec{\mathbf{x}}_2(t) = (\vec{\mathbf{v}}_1 t + \vec{\mathbf{v}}_2)e^{\lambda t}$.

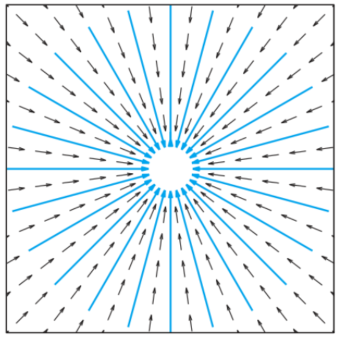
Spring 2018 #19

Phase Portraits 1 Spring 2019 #16 Spring 2018 #18

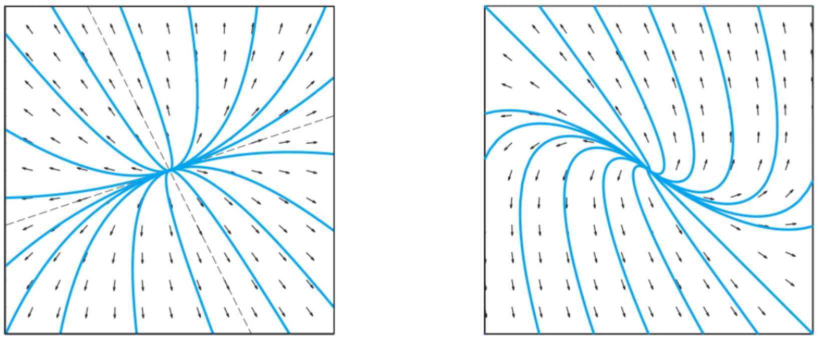
Gallery of Typical Phase Portraits for the System $\mathbf{x}' = \mathbf{A}\mathbf{x}$: Nodes



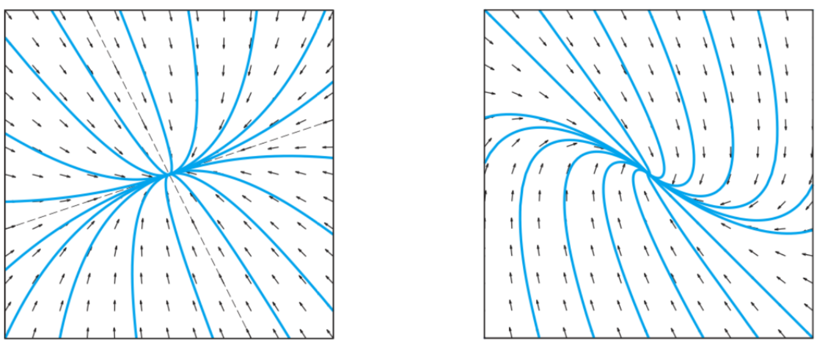
Proper Nodal Source: A repeated positive real eigenvalue with two linearly independent eigenvectors.



Proper Nodal Sink: A repeated negative real eigenvalue with two linearly independent eigenvectors.



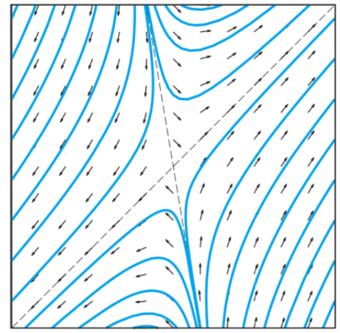
Improper Nodal Source: Distinct positive real eigenvalues (left) or a repeated positive real eigenvalue without two linearly independent eigenvectors (right).



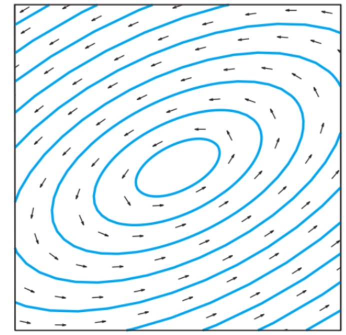
Improper Nodal Sink: Distinct negative real eigenvalues (left) or a repeated negative real eigenvalue without two linearly independent eigenvectors (right).

Phase Portraits 2

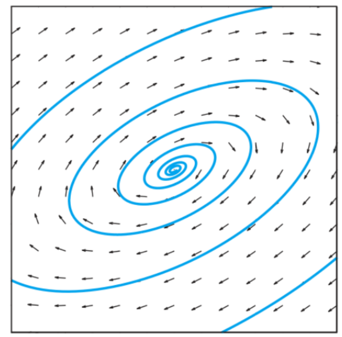
Gallery of Typical Phase Portraits for the System $\mathbf{x}' = \mathbf{A}\mathbf{x}$: Saddles, Centers, Spirals, and Parallel Lines



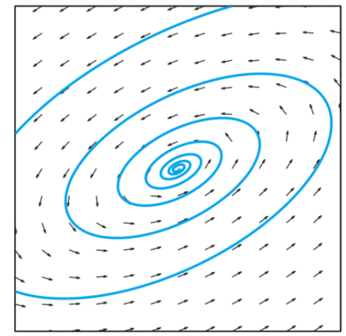
Saddle Point: Real eigenvalues of opposite sign.



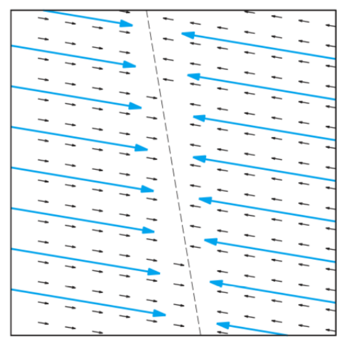
Center: Pure imaginary eigenvalues.



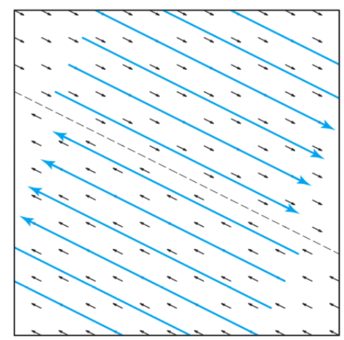
Spiral Source: Complex conjugate eigenvalues with positive real part.



Spiral Sink: Complex conjugate eigenvalues with negative real part.



Parallel Lines: One zero and one negative real eigenvalue. (If the nonzero eigenvalue is positive, then the trajectories flow *away* from the dotted line.)



Parallel Lines: A repeated zero eigenvalue without two linearly independent eigenvectors.

Matrix Exponentials and Linear Systems

Fall 2019 # 13

Fundamental Matrix:

$$\mathbf{\Phi}(t) = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \\ | & | & & | \end{bmatrix}$$

where $\vec{\mathbf{x}}_i$ are fundamental solutions to the system $\dfrac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$.

Exponential matrix: $e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots + \frac{\mathbf{A}^n}{n!} + \cdots$,
$e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$

Matrix Exponential Solutions:

Consider

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

then the solution is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0$.

Nonhomogeneous Linear Systems

Spring 2019 # 18
Spring 2018 #20
Fall 2019 #15
Fall 2018 #20

Consider

$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}(t),$$

a general solution $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_c(t) + \vec{\mathbf{x}}_p(t)$.

Undetermined Coefficients

If $\vec{\mathbf{f}}(t)$ is a linear combination (with constant vector coefficients) of products of polynomials, exponential functions, and sines and cosines. We can make a guess to the general form of a particular solution $\vec{\mathbf{x}}_p$.

See illustrative examples from Lecture Notes Section 5.7.

Variation of Parameters

- Consider

  $$\vec{\mathbf{x}}' = \mathbf{P}(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t),$$

  Then a particular solution is given by

  $$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)dt,$$

  where $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

- In particular, for the initial value problem

  $$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}(t), \quad \vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$$

  Then the solution is given by

  $$\vec{\mathbf{x}}(t) = e^{\mathbf{A}t}\vec{\mathbf{x}}_0 + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}(s)}\vec{\mathbf{f}}(s)ds$$

  Recall $e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$.

- Recall the inverse of 2×2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Laplace Transforms

Definition: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Properties:

- Transform of derivatives:

$$\mathcal{L}\{x\} = X, \quad \mathcal{L}\{x'\} = sX - x(0)$$
$$\mathcal{L}\{x''\} = s^2X - sx(0) - x'(0)$$
$$\mathcal{L}\{x'''\} = s^3X - s^2x(0) - sx'(0) - x''(0)$$

- Transforms of Integrals: $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$
- Translation on the s-Axis: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
- Differentiation of Transforms: Fall 2018 # 18
  $\mathcal{L}\{-tf(t)\} = F'(s)$ and $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \quad n = 1, 2, 3, \ldots$
- Integration of Transforms: $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)d\sigma$
- Translation on the t-Axis: $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$

Spring 2019 #10,#15
Spring 2018 #15 #17
Fall 2019 # 17
Fall 2018 #14
Fall 2018 #17

- Laplace Transform of $\delta(t-c)$: $\mathcal{L}\{\delta(t-c)\} = e^{-cs} \quad (c \geq 0)$

Spring 2019 # 11 #14 Fall 2019 #20
Fall 2018 #16

- Convolutions:
  - Definition: $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$
  - Property: $\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

Spring 2019 #12
Spring 2018 #16
Fall 2019 #19
Fall 2018 #15
Fall 2017 #17

Laplace Transforms of Basic Functions

The following is the usual table of Laplace transforms provided at the end of the exam.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$t^p \ (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\frac{a}{s^2+a^2}$
6.	$\cos at$	$\frac{s}{s^2+a^2}$
7.	$\sinh at$	$\frac{a}{s^2-a^2}$
8.	$\cosh at$	$\frac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$ $= u(t-c)$	$\frac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \ c > 0$
16.	$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17.	$\delta(t-c)$ $= \delta_c(t)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$