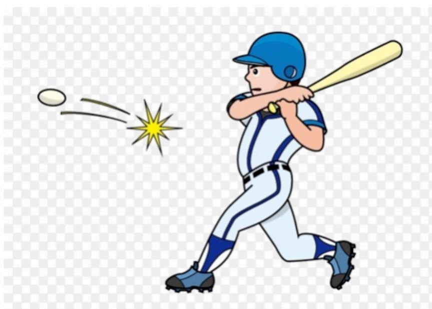
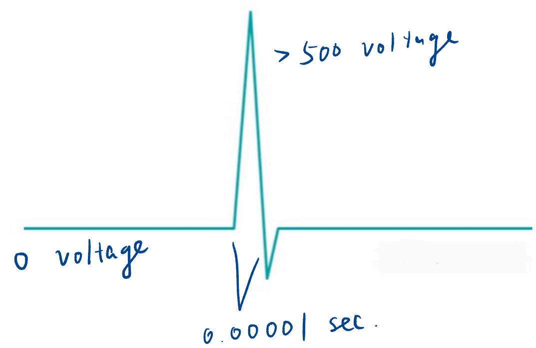
7.6 Impulses and Delta Functions

Introduction of delta function

Impulsive Force

- Consider a force $f(t)$ that acts only during a very short time interval $a \leqq t \leqq b$, with $f(t) = 0$ outside this interval.
- A typical example would be the **impulsive force** of a bat striking a ball---the impact is almost instantaneous.
- A quick surge of voltage (resulting from a lightning bolt, for instance) is an analogous electrical phenomenon.





- In such a situation it often happens that the principal effect of the force depends only on the value of the integral

$$p = \int_a^b f(t)\, dt$$

and does not depend otherwise on precisely how $f(t)$ varies with time t.

- The number p is called the **impulse** of the force $f(t)$ over the interval $[a, b]$.
- In the case of a force $f(t)$ that acts on a particle of mass m in linear motion, integration of Newton's law

$$f(t) = mv'(t) = \tfrac{d}{dt}[mv(t)]$$

yields

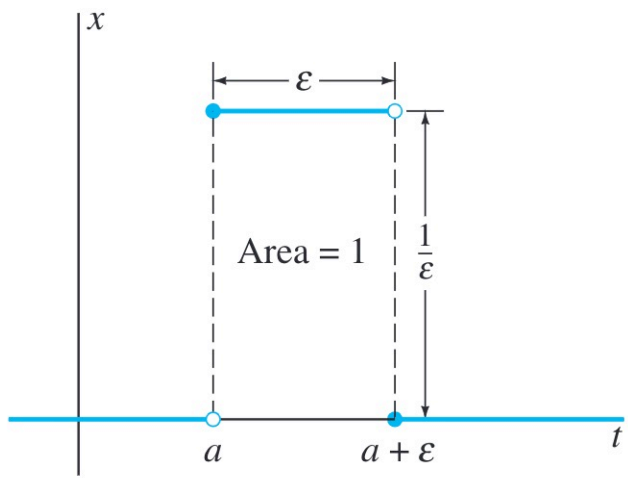
$$p = \int_a^b \frac{d}{dt}[mv(t)]\; dt = mv(b) - mv(a).$$

- Thus the impulse of the force is equal to the change in momentum of the particle.
- We need know neither the precise function $f(t)$ nor even the precise time interval during which it acts.

Modeling Impulsive Forces

- Our strategy for handling such a situation is to set up a reasonable mathematical model in which the unknown force $f(t)$ is replaced with a simple and explicit force that has the same impulse.
- Suppose for simplicity that $f(t)$ has impulse 1 and acts during some brief time interval beginning at time $t = a \geq 0$.
- Then we can select a fixed number $\epsilon > 0$ that approximates the length of this time interval and replace $f(t)$ with the specific function

$$d_{a,\epsilon}(t) = \begin{cases} \dfrac{1}{\epsilon} & \text{if } a \leq t < a + \epsilon, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$



- If $b \geq a + \epsilon$, then as the figure shows, the impulse of $d_{a,\epsilon}$ over $[a, b]$ is

$$p = \int_a^b d_{a,\epsilon}(t)\,dt = \int_a^{a+\epsilon} \frac{1}{\epsilon}\,dt = 1.$$

- Because the precise time interval during which the force acts seems unimportant, it is tempting to think of an **instantaneous impulse** that occurs precisely at the instant $t = a$.
- We might try to model such an instantaneous unit impulse by taking the limit as $\epsilon \to 0$, thereby defining

$$\delta_a(t) = \lim_{\epsilon \to 0} d_{a,\epsilon}(t), \text{ where } a \geq 0. \tag{1}$$

Dirac Delta

- If we could also take the limit under the integral sign in the equation

$$\int_0^\infty d_{a,\epsilon}(t)\,dt = 1, \tag{2}$$

  then it would follow that

$$\int_0^\infty \delta_a(t)\,dt = 1. \tag{3}$$

- But the limit in the equation

$$\delta_a(t) = \lim_{\epsilon\to 0} d_{a,\epsilon}(t), \tag{4}$$

  gives

$$\delta_a(t) = \begin{cases} +\infty & \text{if } t = a, \\ 0 & \text{if } t \neq a. \end{cases} \tag{5}$$

- Obviously, no actual function can satisfy both of these conditions---if a function is zero except at a single point, then its integral is not 1 but zero.
- Nevertheless, the symbol $\delta_a(t)$ is very useful.
- However interpreted, it is called the **Dirac delta function** at a after the British theoretical physicist Dirac (1902--1984), who in the early 1930s introduced a "function'' with the above properties.

Delta Functions as Operators

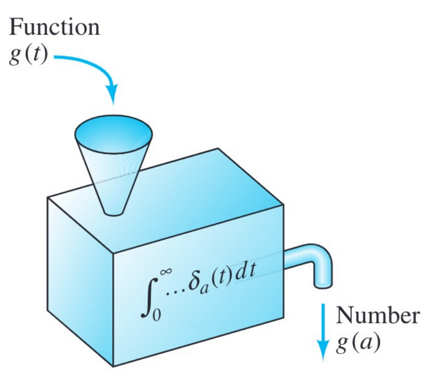
Definition of $\delta_a(t)$. We take the following equation as the definition of the symbol $\delta_a(t)$.

$$\int_0^\infty g(t)\delta_a(t)\,dt = g(a). \tag{6}$$

Remark. Although we call it the delta function, it is not a function; instead, it specifies the

$$\int_0^\infty \cdots \delta_a(t)\,dt, \tag{7}$$

which---when applied to a continuous function $g(t)$---sifts out or selects the value $g(a)$ of this function at the point $a \geq 0$.



Laplace Transform of $\delta_a(t)$

If we take $g(t) = e^{-st}$ in our definition of $\delta_a(t)$, the result is

$$\int_0^\infty e^{-st}\delta_a(t)\,dt = e^{-as}. \tag{8}$$

We therefore *define* the Laplace transform of the delta function to be

$$\mathcal{L}\{\delta_a(t)\} = e^{-as} \quad (a \geq 0). \tag{9}$$

We write

$$\delta(t) = \delta_0(t) \quad \text{and} \quad \delta(t-a) = \delta_a(t), \tag{10}$$

then

$$\mathcal{L}\{\delta(t)\} = 1 \qquad \mathcal{L}\{\delta(t-a)\} = e^{-as} \quad (a \geq 0) \tag{11}$$

Example 1 Solve the initial value problem and graph the solution function $x(t)$.

$$x'' + 4x' + 4x = 1 + \delta(t-2); \quad x(0) = x'(0) = 0. \tag{12}$$

ANS: We apply the Laplace transform on both sides of Eq (12)

Note

$$\mathcal{L}\{x''\} = s^2 X(s) - s\cancel{x(0)}^{0} - \cancel{x'(0)}^{0}$$

$$\mathcal{L}\{x'\} = sX(s) - \cancel{x(0)}^{0}$$

$$s^2 X + 4s X(s) + 4X(s) = \frac{1}{s} + e^{-2s}$$

$$\Rightarrow (s^2 + 4s + 4) X(s) = \frac{1}{s} + e^{-2s}$$

$$\Rightarrow X(s) = \frac{1}{s(s+2)^2} + \frac{e^{-2s}}{(s+2)^2}$$

Assume $\displaystyle\frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$

$$= \frac{A(s+2)^2 + B\cdot s(s+2) + C\cdot s}{s(s+2)^2}$$

$$= \frac{4A + (4A+2B+C)\cdot s + (A+B)s^2}{s(s+2)^2}$$

$$\begin{cases} 4A = 1 \\ A+B = 0 \\ 4A+2B+C = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \\ C = -\frac{1}{2} \end{cases}$$

Thus $\displaystyle X(s) = \frac{1}{4}\left[\frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}\right] + \frac{e^{-2s}}{(s+2)^2}$

$$\mathcal{L}^{-1}\{X(s)\} = \frac{1}{4}\left[\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \underbrace{\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2}\right\}}_{①}\right] + \underbrace{\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s+2)^2}\right\}}_{②}$$

Note ① $\displaystyle\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2}\right\} = 2e^{-2t}\cdot t$

Recall $\mathcal{L}\{e^{at} \cdot t^n\} = \dfrac{n!}{(s-a)^{n+1}} \quad (s>a)$

② $$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s+2)^2}\right\} \xrightarrow[a=2]{F(s)=\frac{1}{(s+2)^2} \Rightarrow f(t)=e^{-2t}\cdot t} u(t-2)\cdot f(t-2)$$

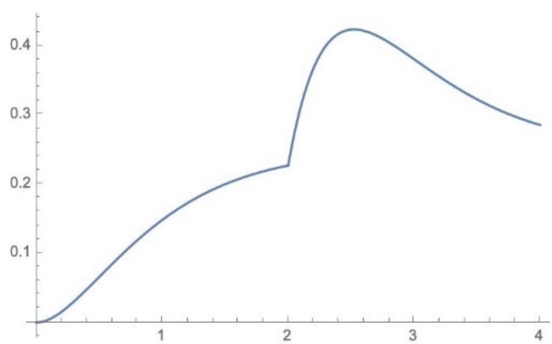
$$= u(t-2)\cdot e^{-2(t-2)}\cdot(t-2)$$

Recall $\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\, f(t-a)$

Let $F(s) = \dfrac{1}{(s+2)^2}$, then $f(t) = e^{-2t}\cdot t$

Thus

$$x(t) = \frac{1}{4}\left[1 - e^{-2s} - 2te^{-2t}\right] + u(t-2)\cdot(t-2)e^{-2(t-2)}$$



Recall $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

Example 2 Solve the initial value problem and graph the solution function $x(t)$.

$$x'' + 2x' + 2x = 2\delta(t-\pi); \qquad x(0) = x'(0) = 0 \tag{13}$$

Ans: We apply the Laplace transform on both sides of Eq (13)

$$s^2 X(s) + 2s\,X(s) + 2X(s) = 2 \cdot e^{-\pi s}$$

$$\Rightarrow \quad X(s) = \frac{2e^{-\pi s}}{s^2+2s+2} = \frac{2e^{-\pi s}}{(s+1)^2+1}$$

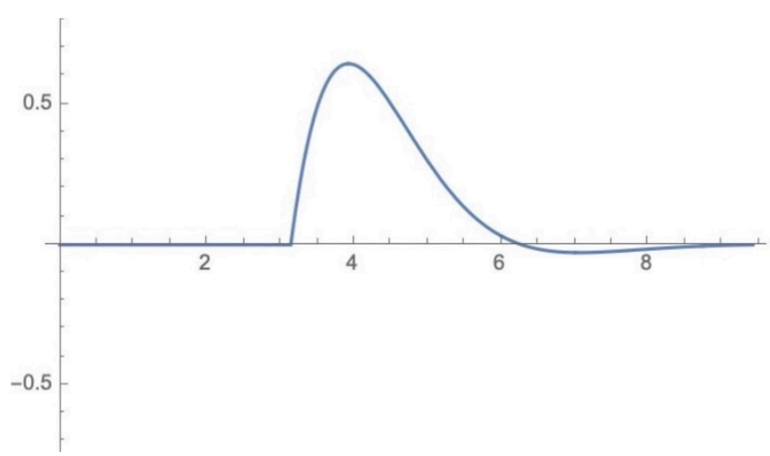
Recall $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t} \cdot \sin t$

Thus $\mathcal{L}^{-1}\left\{\frac{2e^{-\pi s}}{(s+1)^2+1}\right\} \xlongequal{f(t)=e^{-t}\sin t} 2 \cdot u(t-\pi)\, e^{-(t-\pi)} \sin(t-\pi)$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\, f(t-a)$

$$= \begin{cases} 0, & 0 \le t \le \pi \\ -2e^{-(t-\pi)} \sin t, & t > \pi \end{cases}$$

$\sin(t-\pi) = -\sin t$



Systems Analysis and Duhamel's Principle

Consider a physical system, like mass-spring-dashpot system and the series RLC circuit , described by the differential equation

$$ax'' + bx' + cx = f(t). \tag{14}$$

The constant coefficients a, b, and c are determined by the physical parameters of the system and are independent of $f(t)$.

For simplicity we assume that the system is initially passive: $x(0) = x'(0) = 0$. Then the transform of our differential equation is

$$as^2X(s) + bsX(s) + cX(s) = F(s), \tag{15}$$

so

$$X(s) = \frac{F(s)}{as^2 + bs + c} = W(s)F(s). \tag{16}$$

The function

$$W(s) = \frac{1}{as^2 + bs + c} \tag{17}$$

is called the **transfer function** of the system.
The function

$$w(t) = \mathcal{L}^{-1}\{W(s)\} \tag{18}$$

is called the **weight function** of the system.
From the fact that $X(s) = W(s)F(s)$ we see by convolution that

$$x(t) = \int_0^t w(\tau)f(t-\tau)\,d\tau. \;=\; w(t) * f(t) \tag{19}$$

This formula is **Duhamel's principle** for the system.

Example 3 Apply Duhamel's principle to write an integral formula for the solution of the initial value problem.

$$x'' + 6x' + 9x = f(t); \quad x(0) = x(0)' = 0 \tag{20}$$

ANS: Apply the Laplace transform. on both sides of the Eq 20.

$$s^2 X(s) + 6s X(s) + 9 X(s) = F(s)$$

$$\Rightarrow \quad X(s) = \underbrace{\frac{1}{(s+3)^2}}_{W(s)} F(s)$$

Apply Duhamel's principle, we know

$$W(s) = \frac{1}{(s+3)^2}$$

$$w(t) = \mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} = e^{-3t} \cdot t$$

Thus

$$x(t) = w(t) * f(t)$$

$$= \int_0^t w(\tau) f(t-\tau)\, d\tau$$

$$= \int_0^t e^{-3\tau} \cdot \tau \; f(t-\tau)\, d\tau$$