Thursday:

- Review: common ODE. pdf.

  Brightspace → Content → Useful links

- Review final from Spring 2017.

Next week:

Additional Office hours: Check in the announcement.

Dec. 16.

7.5 Periodic and Piecewise Continuous Input Functions

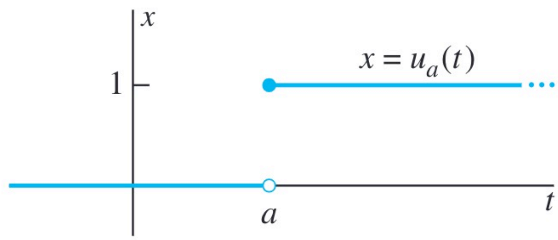
Unit Step Function

The unit step function at $t = a$ is defined by

$$u_a(t) = u(t-a) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t \geq a. \end{cases} \tag{1}$$

and if $a = 0$,

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases} \tag{2}$$



If $a > 0$, then

by def

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u_a(t) dt = \int_a^\infty e^{-st} dt = \lim_{b\to\infty} \left[-\frac{e^{-st}}{s} \right]_{t=a}^{b}.$$

Thus

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}, \quad (s > 0,\ a > 0). \tag{3}$$

- Recall $\mathcal{L}\{u(t)\} = \frac{1}{s}$, this equation implies that multiplication of the transform of $u(t)$ by e^{-as} corresponds to the translation $t \to t - a$ in the original independent variable.
- Theorem 1 tells us that this fact, when properly interpreted, is a general property of the Laplace transformation.

Theorem 1 Translation on the t-Axis

If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \tag{4}$$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a) \tag{5}$$

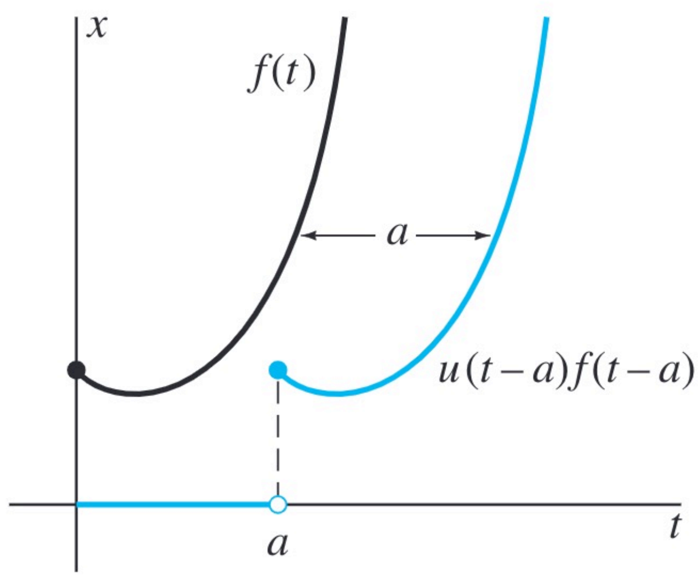
for $s > c + a$.

Remark

- Note that

$$u(t-a)f(t-a) = \begin{cases} 0 & \text{if } t < a, \\ f(t-a) & \text{if } t \geq a. \end{cases} \tag{6}$$

- Thus Theorem 1 implies that $\mathcal{L}^{-1}\{e^{-as}F(s)\}$ is the function whose graph for $t \geq a$ is the translation by a units to the right of the graph of $f(t)$ for $t \geq 0$.



We will use Eq(5) to find the inverse Laplace transform of Example 1 and Example 2.

Recall

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a) \tag{5}$$

Example 1 Find the inverse Laplace transform f of the given functions. Then sketch the graph of f.

$$F(s) = \frac{e^{-3s}}{s^2}$$

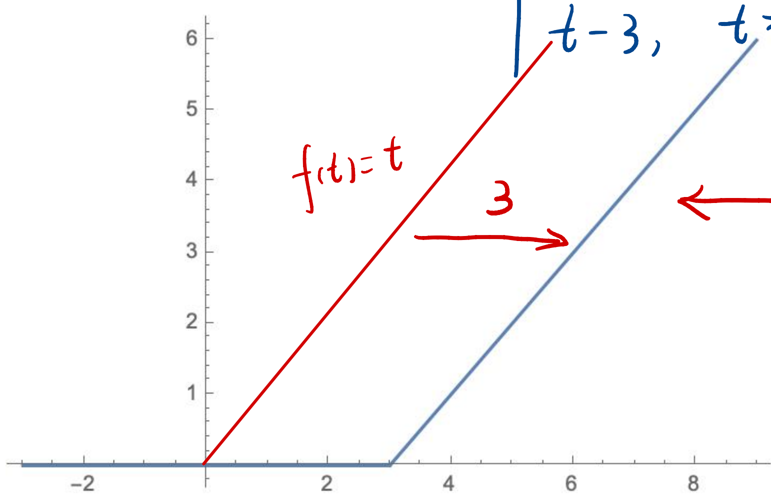
$a=3$ $\quad F(s) = \frac{1}{s^2} \Rightarrow f(t) = t$

ANS: By Eq(5). $\quad \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} = u(t-3)\cdot f(t-3)$

$$= u(t-3)\cdot(t-3)$$

$$= \left(\begin{cases} 0, & t<3 \\ 1, & t \geq 3 \end{cases}\right)\cdot(t-3)$$

$$= \begin{cases} 0, & t<3 \\ t-3, & t \geq 3 \end{cases}$$



$\leftarrow [u(t-3)](t-3)$

Example 2 Find the inverse Laplace transform f of the given functions. Then sketch the graph of f.

$$F(s) = \frac{s(1+e^{-3s})}{s^2+\pi^2}$$

ANS:

$$= \frac{s}{s^2+\pi^2} + e^{-3s}\frac{s}{s^2+\pi^2}.$$

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)}$$

$$F(s) = \frac{s}{s^2+\pi^2}$$

$$f(t) = \cos\pi t$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} + \mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{s}{s^2+\pi^2}\right\}$$

$$= \cos\pi t + u(t-3)\cdot f(t-3)$$

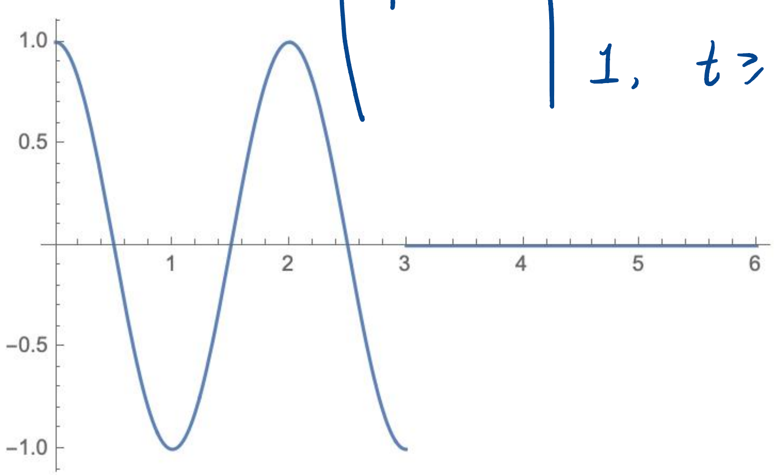
$$= \cos\pi t + u(t-3)\cdot \cos[\pi(t-3)]$$

$$\cos(\pi t - 3\pi) = -\cos\pi t$$

$$= \cos\pi t - u(t-3)\cdot\cos\pi t$$

$$= [1-u(t-3)]\cdot\cos\pi t$$

$$= \left(1 - \begin{cases} 0, & t<3 \\ 1, & t\geq 3\end{cases}\right)\cdot\cos\pi t$$

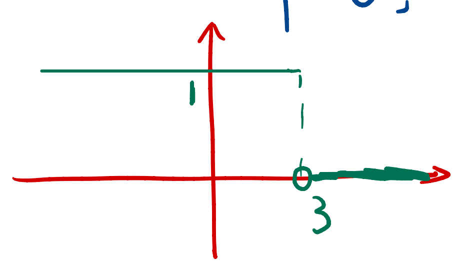


$$= \left(\begin{cases} 1-0=1, & t<3 \\ 1-1=0, & t\geq 3\end{cases}\right)\cdot\cos\pi t$$

$$= \begin{cases} \cos\pi t, & t<3 \\ 0, & t\geq 3\end{cases}$$

Note

$$1-u(t-3) = \begin{cases} 1, & t<3 \\ 0, & t\geq 3\end{cases}$$



We will use Eq (6) to find the Laplace transform of Example 3 and Example 4.

Recall

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \tag{6}$$

Example 3 Find the Laplace transform of the given function.

$$f(t) = \begin{cases} \cos 2t & \text{if } 0 \le t < 2\pi, \\ 0 & \text{if } t \ge 2\pi. \end{cases} \tag{7}$$

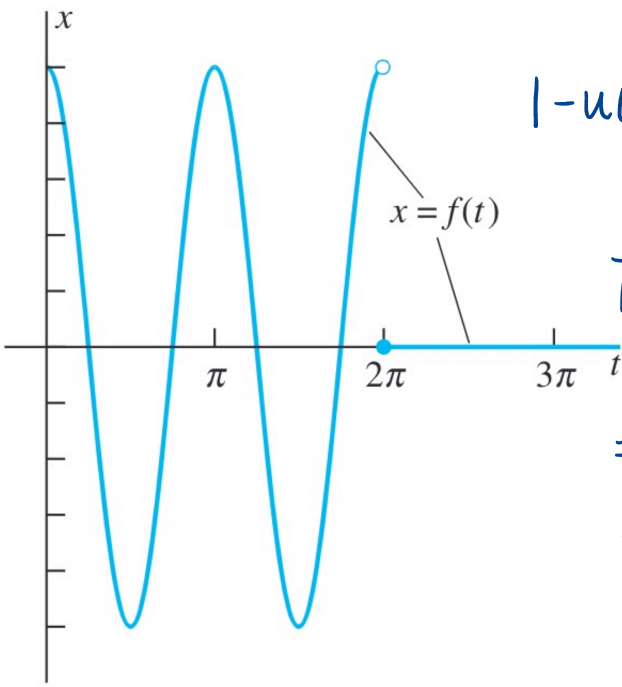
The graph of $f(t)$ is shown in the figure.

ANS: We need to write $f(t)$ in terms of $u(t-a)$ & $g(t-a)$ when $t \geqslant 0$.

$$u(t-2\pi) = \begin{cases} 0, & 0 \le t < 2\pi \\ 1, & t \geqslant 2\pi \end{cases}$$

$$\Downarrow$$

$$1 - u(t-2\pi) = \begin{cases} 1, & 0 \le t < 2\pi \\ 0, & t \geqslant 2\pi \end{cases}$$



Then $f(t) = [1 - u(t-2\pi)] \cdot \cos 2t$

$= \cos 2t - u(t-2\pi) \cdot \cos 2t$

$= \cos 2t - u(t-2\pi) \cdot \underbrace{\cos[2(t-2\pi)]}_{\cos(2t-4\pi) = \cos 2t}$

Then $\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos 2t\} - \mathcal{L}\{u(t-2\pi) \cdot \cos[2(t-2\pi)]\}$

$\Rightarrow f(t-2\pi) = \cos[2(t-2\pi)]$

$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$

$\Downarrow$

$f(t) = \cos 2t$

$$= \frac{s}{s^2+4} - e^{-2\pi s} \cdot \frac{s}{s^2+4}$$

$$= s\,\frac{(1-e^{-2\pi s})}{s^2+4}$$

Example 4 Find the Laplace transform of the given function.

$$f(t) = \begin{cases} \sin t & \text{if } 0 \le t \le 3\pi, \\ 0 & \text{if } t > 3\pi. \end{cases} \tag{8}$$

ANS: $f(t) = [1 - u(t-3\pi)] \cdot \sin t$

want $t-3\pi$

$= \sin t - u(t-3\pi) \cdot \sin t$

Recall $\sin(t-3\pi) = -\sin t$

$= \sin t + u(t-3\pi) \cdot \sin(t-3\pi)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{u(t-3\pi) \cdot \underbrace{\sin(t-3\pi)}_{f(t-3\pi)}\}$

$\Downarrow$

$f(t) = \sin t$

$$= \frac{1}{s^2+1} + e^{-3\pi s} \cdot \frac{1}{s^2+1}$$

$$= \frac{1+e^{-3\pi s}}{s^2+1}$$

Reading material

Transforms of Periodic Functions

Periodic Forcing Functions

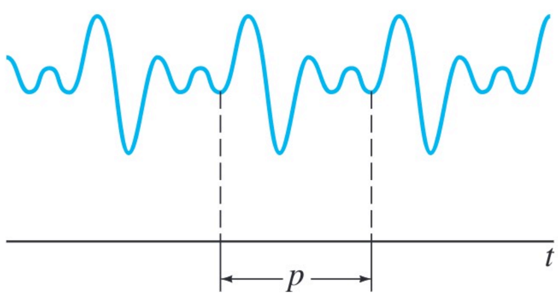
- Periodic forcing functions in practical mechanical or electrical systems often are more complicated than pure sines or cosines.
- The nonconstant function $f(t)$ defined for $t \geq 0$ is said to be **periodic** if there is a number $p > 0$ such that

$$f(t+p) = f(t) \tag{9}$$

  for all $t \geq 0$.
- The least positive value of p (if any) for which this equation holds is called the **period** of f.

Periodic Functions

- Such a function is shown in the figure.



- Theorem 2 simplifies the computation of the Laplace transform of a periodic function.

Theorem 2 Transforms of Periodic Functions
Let $f(t)$ be periodic with period p and piecewise continuous for $t \geq 0$.
Then the transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > 0$ and is given by

$$F(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt. \tag{10}$$