7.4 Derivatives, Integrals, and Products of Transforms

1. Products of Transforms

Consider the initial value problem

$$x'' + x = \cos t; \quad x(0) = x'(0) = 0$$

We apply the Laplace transform on both sides of the equation,

$$\mathcal{L}\{x''\} + \mathcal{L}\{x\} = \mathcal{L}\{\cos t\}$$

Recall $\mathcal{L}\{x''\} = s^2\mathcal{L}\{x\} - sx(0) - x'(0) = s^2X(s)$ and $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$.

We have $(s^2+1)X(s) = \dfrac{s}{s^2+1}$ thus

$$X(s) = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1} = \mathcal{L}\{\cos t\} \cdot \mathcal{L}\{\sin t\}$$

Question 1: Do we have $\mathcal{L}\{\cos t\} \cdot \mathcal{L}\{\sin t\} = \mathcal{L}\{\cos t \sin t\}$? The answer is no, since

$$\mathcal{L}\{\cos t \sin t\} = \mathcal{L}\{\frac{1}{2}\sin 2t\} = \frac{1}{s^2+4} \neq \frac{s}{s^2+1} \cdot \frac{1}{s^2+1}.$$

Question 2:

If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, what is $\mathcal{L}^{-1}\{F(s) \cdot G(s)\}$?

Theorem 1 tells us the answer is the following function

$$\int_0^t f(\tau)g(t-\tau)\,d\tau.$$

We call this function the convolution of f and g and it is denoted as $f * g$.

Definition. The Convolution of Two Functions

The **convolution** $f * g$ of the piecewise continuous functions f and g is defined for $t \geq 0$ as follows:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau \qquad (1)$$

We will also write $f(t) * g(t)$ when convenient.

Remark: The convolution is commutative: $f*g = g*f$

If we substitute $u = t - \tau$ in (1), $0 \leq \tau \leq t$, $\tau = 0,\ u = t$; $\tau = t,\ u = 0$

$\Rightarrow \tau = t - u$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_t^0 f(t-u)\,g(u)\,d(-u) = \int_0^t g(u)f(t-u)\,du$$

$= (g * f)(t)$

Example 1 Find the convolution $f(t) * g(t)$ in the given problem

$f(t) = \cos t, \quad g(t) = \sin t$

ANS: By Eq (1). $(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau$

Thus $(\cos t) * (\sin t) = \int_0^t \cos\tau \sin(t-\tau)\, d\tau$

Recall $\cos A \cdot \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

Then $\int_0^t \cos\tau \sin(t-\tau) d\tau$

constant for τ

$= \frac{1}{2} \int_0^t [\sin t - \sin(2\tau - t)] d\tau$

$\int \sin(2\tau - t) d\tau$

$= \frac{1}{2}\int \sin(2\tau - t)\, d(2\tau - t)$

$= -\frac{1}{2}\cos(2\tau - t)$

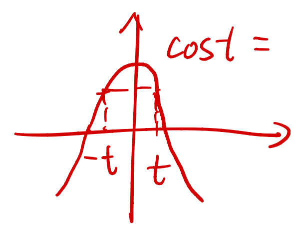
$= \frac{1}{2}\left[\tau \sin t + \frac{1}{2}\cos(2\tau - t)\right]\Big|_0^t$

$= \frac{1}{2}\left[t\sin t + \cancel{\frac{1}{2}\cos t} - 0 - \cancel{\frac{1}{2}\cos(-t)}\right]$

Recall

$\cos(-t) = \cos t$

$\cos t = \cos(-t)$



$= \frac{1}{2} t \sin t$

Theorem 1 The Convolution Property
Suppose that $f(t)$ and $g(t)$ are piecewise continuous for $t \geqq 0$ and that $|f(t)|$ and $|g(t)|$ are bounded by Me^{ct} as $t \to +\infty$. Then the Laplace transform of the convolution $f(t) * g(t)$ exists for $s > c$; moreover,

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$$

and

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t).$$

Finding Inverse Transforms

Thus we can find the inverse transform of the product $F(s) \cdot G(s)$, provided that we can evaluate the integral

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(\tau) g(t-\tau)\, d\tau.$$

$$= f * g$$

Example 2 Apply the convolution theorem to find the inverse Laplace transform of the function.

$$H(s) = \frac{2}{(s-1)(s^2+4)} = \frac{2}{s^2+2^2} \cdot \frac{1}{s-1}$$

ANS:

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2} \cdot \frac{1}{s-1}\right\}$$

Recall $\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \sin 2t$, $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$

Let $f(t) = \sin 2t$, and $g(t) = e^t$.

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-1)(s^2+4)}\right\} = f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$\to e^{t-\tau} = e^t \cdot e^{-\tau}$$

$$= \int_0^t \sin 2\tau \, e^{t-\tau}\, d\tau = e^t \int_0^t \sin 2\tau \, e^{-\tau}\, d\tau$$

$$\Rightarrow \begin{cases} a = 2 \\ b = -1 \end{cases}$$

Note: $\int e^{bx} \sin ax \, dx = \frac{1}{a^2+b^2} e^{bx} (b \sin ax - a \cos ax)$

$$= e^t \frac{1}{2^2+(-1)^2} e^{-\tau} (-\sin 2\tau - 2\cos 2\tau) \Big|_0^t$$

$$= e^t \cdot \frac{1}{5} \left[e^{-t} \cdot (-\sin 2t - 2\cos 2t) - e^0 (-0-2) \right]$$

$$= \frac{1}{5}(-\sin 2t - 2\cos 2t) + \frac{2}{5} e^t$$

$$= \frac{2}{5} e^t - \frac{1}{5} \sin 2t - \frac{2}{5} \cos 2t$$

$$\Rightarrow \mathcal{L}^{-1}\{H(s)\} = \frac{2}{5}e^{t} - \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

2. Differentiation of Transforms

Question 3: What is $F'(s)$ if $\mathcal{L}\{f(t)\} = F(s)$?

Theorem 2

If $f(t)$ is piecewise continuous for $t \geq 0$ and $|f(t)| \leq Me^{ct}$ as $t \to +\infty$, then

$$\mathcal{L}\{-tf(t)\} = F'(s)$$

for $s > c$. Equivalently,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}.$$

Repeated application of Equation (7) gives

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \quad n = 1, 2, 3, \ldots \quad \otimes$$

Example 3 Apply Theorem 2 to find the Laplace transform of $f(t)$.

(1) $f(t) = t^2 \cos kt$ (Exercise)

(2) $f(t) = te^{-t}\sin 2t$ (Exercise)

(3) $f(t) = t^2 \sin kt$

ANS: By Eq $\otimes$, $\mathcal{L}\{t^2 \sin kt\} = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{k}{s^2+k^2}\right)$

$$= \frac{d}{ds}\left(\frac{-2s\cdot k}{(s^2+k^2)^2}\right) \qquad \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

$$= \frac{-2k(s^2+k^2)^2 - [(s^2+k^2)^2]'\cdot(-2ks)}{(s^2+k^2)^4}$$

$$= \frac{-2k(s^2+k^2)^2 - 2\cdot(s^2+k^2)\cdot 2s\cdot(-2ks)}{(s^2+k^2)^4}$$

$$= \frac{-2k(s^2+k^2)^{\cancel{2}\,1} + 8ks^2\,\cancel{(s^2+k^2)}\,1}{(s^2+k^2)^{\cancel{4}\,3}}$$

$$= \frac{-2k(s^2+k^2) + 8ks^2}{(s^2+k^2)^3} = \frac{6ks^2 - 2k^3}{(s^2+k^2)^3}$$

(1) By Eq $\circledast$

$$\mathcal{L}\{t^2\cos 2t\} = (-1)^2 \frac{d^2}{ds^2} \frac{s}{s^2+2^2}$$

$$= \frac{d}{ds}\left(\frac{s^2+2^2-2s\cdot s}{(s^2+2^2)^2}\right) = \frac{d}{ds}\left(\frac{-s^2+2^2}{(s^2+2^2)^2}\right)$$

$$= \frac{-2s\cdot(s^2+2^2)^2-\left[(s^2+2^2)^2\right]'\cdot(-s^2+4)}{(s^2+2^2)^4}$$

$$= \frac{-2s(s^2+2^2)^2-2(s^2+4)\cdot 2s\cdot(-s^2+4)}{(s^2+2^2)^4}$$

$$= \frac{-2s(s^2+2^2)^2-4s(s^2+4)(-s^2+4)}{(s^2+4)^4}$$

$$= \frac{-2s(s^2+2^2)-4s(-s^2+4)}{(s^2+4)^3}$$

$$= \frac{-24s}{(s^2+4)^3} + \frac{2s^3}{(s^2+4)^3}$$

(2) By Eq $\circledast$ and $\mathcal{L}\{e^{-t}\sin 2t\} = \frac{2}{(s+1)^2+2^2} = \frac{2}{s^2+2s+5}$

$$\mathcal{L}\{t\,e^{-t}\sin 2t\}$$

$$= (-1)\frac{d}{ds}\cdot\frac{2}{s^2+2s+5}$$

$$= +\frac{2(s^2+2s-5)'}{(s^2+2s+5)^2}$$

$$= \frac{4(s+1)}{(s^2+2s+5)^2}$$

3. Integration of Transforms

- In Theorem 2, $F'(s)$ corresponds to multiplication of $f(t)$ by t (together with a change of sign).
- It is therefore natural to expect that integration of $F(s)$ will correspond to division of $f(t)$ by t (Theorem 3).

Theorem 3. Integration of Transforms
Suppose that $f(t)$ is piecewise continuous for $t \geq 0$, that $f(t)$ satisfies the condition

$$\lim_{t\to 0^+} \frac{f(t)}{t} \quad \text{exists and is finite,} \qquad (1)$$

and that $|f(t)| \leq Me^{ct}$ as $t \to +\infty$.
Then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\, d\sigma$$

for $s > c$. Equivalently,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\, d\sigma\right\}.$$

L'Hôpital's rule states that for functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \text{ or } \pm\infty$, and $g'(x) \neq 0$ for all x in I with $x \neq c$, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

The differentiation of the numerator and denominator often simplifies the quotient or converts it to a limit that can be evaluated directly.

Example 5 Apply Theorem 3 to find the Laplace transform of $f(t)$.

$$f(t) = \frac{\sinh t}{t}$$

ANS: $\mathcal{L}\{\sinh t\} = \frac{k}{s^2 - k^2} \quad (s > |k|) \quad s > 1$

We first verify the condition (1) holds.

$$\lim_{t\to 0^+} \frac{\sinh t}{t} = \lim_{t\to 0^+} \frac{e^t - e^{-t}}{2\cdot t} \ \underset{\text{"}\frac{0}{0}\text{"}}{\overset{\text{L'Hospital}}{=}} \ \lim_{t\to 0^+} \frac{(e^t - e^{-t})'}{(2t)'}$$

$$= \lim_{t\to 0^+} \frac{e^t + e^{-t}}{2} = 1 \quad \text{(exists \& finite)}$$

Then by Thm 3.

$$\mathcal{L}\left\{\frac{\sinh t}{t}\right\} = \int_s^\infty \frac{1}{\sigma^2 - 1}\, d\sigma = \int_s^\infty \frac{1}{(\sigma+1)(\sigma-1)}\, d\sigma$$

$$\overset{\text{Partial fraction method}}{=} \frac{1}{2}\int_s^\infty \left(\frac{1}{\sigma - 1} - \frac{1}{\sigma + 1}\right) d\sigma$$

$$= \frac{1}{2}\left[\ln|\sigma-1| - \ln|\sigma+1|\right]_s^{\infty} \qquad \ln x - \ln y = \ln\frac{x}{y}$$

$$= \frac{1}{2}\ln\left|\frac{\sigma-1}{\sigma+1}\right|\Bigg|_s^{\infty}$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\cancel{\ln\left|\frac{b-1}{b+1}\right|}^{\,0} - \ln\left|\frac{s-1}{s+1}\right|\right]$$

Note $\lim_{b\to\infty}\ln\left|\frac{b-1}{b+1}\right| = \ln\lim_{b\to\infty}\left|\frac{b-1}{b+1}\right| = \ln\lim_{b\to\infty}\left|\frac{b+1-2}{b+1}\right|$

$$= \ln\lim_{b\to\infty}\left|1-\cancel{\frac{2}{b+1}}^{\,0}\right| = \ln 1 = 0$$

$$= \frac{1}{2}\left(-\ln\left|\frac{s-1}{s+1}\right|\right) = \frac{1}{2}\left(\ln\left|\frac{s-1}{s+1}\right|^{-1}\right) = \frac{1}{2}\ln\left|\frac{s+1}{s-1}\right| = \frac{1}{2}\ln\frac{s+1}{s-1}$$

Example 6 Apply the convolution theorem to derive the indicated solution $x(t)$ of the given differential equation with initial conditions $x(0) = x'(0) = 0.$

$$x'' + 4x = f(t); \quad x(t) = \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\, d\tau$$

ANS: We apply the Laplace transform on both sides of the given eqn.

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow s^2X(s) - \cancel{s\cdot x(0)}^{\,0} - \cancel{x'(0)}^{\,0} + 4X(s) = F(s)$$

$$\Rightarrow (s^2+4)X(s) = F(s)$$

$$\Rightarrow X(s) = \frac{F(s)}{s^2+4} = \underbrace{\frac{1}{s^2+4}}_{G(s)}F(s)$$

Note $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin 2t$

$= g(t)$

$\frac{1}{s^2+4}$

$X(s) = G(s)\cdot F(s)$ Apply

Thus

$x(t) = g(t) * f(t)$

$\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = f(t) * g(t).$

$= \left(\frac{1}{2}\sin 2t\right) * f(t)$

by def of "$*$"

$= \frac{1}{2}\int_0^t \sin 2\tau \, f(t-\tau)\, d\tau$