7.3 Translation and Partial Fractions

Recall in the examples of Section 7.2, the solution of a linear differential equation can be reduced to finding the inverse Laplace transform of a function

$$R(s) = \frac{P(s)}{Q(s)}$$

where the degree of $P(s)$ is less than $Q(s)$.

Question 1: How to decompose $R(s)$?

The following two rules describe the partial fraction decomposition of $R(s)$, in terms of the factorization of the denominator $Q(s)$ into linear factors (rule 1) and irreducible quadratic factors (rule 2).

Rule 1. Linear Factor Partial Fractions

The portion of the partial fraction decomposition of $R(s)$ corresponding to the linear factor $s-a$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n},$$

where A_1, $A_2, \cdots$, and A_n are constants.

Rule 2. Quadratic Factor Partial fractions

The portion of the partial fraction decomposition corresponding to the irreducible quadratic factor $(s-a)^2+b^2$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{[(s-a)^2+b^2]^2} + \cdots + \frac{A_n s + B_n}{[(s-a)^2+b^2]^n},$$

where $A_1, A_2, \cdots, A_n$, $B_1, B_2, \cdots$, and B_n are constants.

Question 2: How to find $F(s-a)$ if $F(s) = \mathcal{L}\{f(t)\}$?

Theorem 1. Translation on the s-Axis

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > c$, then $\mathcal{L}\{e^{at} f(t)\}$ exists for $s > a + c$, and

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

Equivalently,

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t).$$

Thus the translation $s \to s - a$ in the transform corresponds to multiplication of the orginal function of t by e^{at}.

Proof :

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)dt = \int_0^\infty e^{-st}[e^{at} f(t)]dt = \mathcal{L}\{e^{at} f(t)\}.$$

We apply the translation theorem to the formulas for the Laplace transforms of t^n, $\cos kt$ and $\sin kt$, we have

$f(t)$	$F(s)$	
$e^{at} t^n$	$\dfrac{n!}{(s-a)^{n+1}}$	$(s > a)$
$e^{at} \cos kt$	$\dfrac{s-a}{(s-a)^2 + k^2}$	$(s > a)$
$e^{at} \sin kt$	$\dfrac{k}{(s-a)^2 + k^2}$	$(s > a)$

We can also check Table 1 to get the solution

Example 1 Apply the translation theorem to find the Laplace transforms of the functions.

(1) $x(t) = t^3 e^{3t}$ $\quad a=3,\ f(t)=t^3$

(2) $x(t) = e^{-3t} \cos 5\pi t$ $\quad a=-3,\ f(t)=\cos 5\pi t$

ANS: (1) Recall $\mathcal{L}\{t^3\} = \frac{3!}{t^4}$. By Thm 1. $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

We have $\mathcal{L}\{e^{3t} t^3\} = \frac{3!}{(t-3)^4}$

(2) Recall $\mathcal{L}\{\cos 5\pi t\} = \frac{s}{s^2+(5\pi)^2}$.

By Thm 1. $\mathcal{L}\{e^{-3t} \cos 5\pi t\} = \frac{s-(-3)}{(s-(-3))^2+25\pi^2}$

$= \frac{s+3}{(s+3)^2+25\pi^2}$

Example 2 Apply the translation theorem to find the inverse Laplace transform of the function.

$$F(s) = \frac{3s+5}{s^2-6s+25}$$

ANS: $F(s) = \frac{3s+5}{s^2-6s+25} = \frac{3s+5}{s^2-6s+9-9+25} = \frac{3(s-3)+9+5}{(s-3)^2+16}$

$= 3 \cdot \frac{s-3}{(s-3)^2+4^2} + \frac{\cancel{14}^{\,\frac{7}{2}}}{\cancel{4}} \frac{4}{(s-3)^2+4^2}$

$= 3 \cdot \frac{s-3}{(s-3)^2+4^2} + \frac{7}{2} \frac{4}{(s-3)^2+4^2}$

Then by Table 1.

$\mathcal{L}^{-1}\{F(s)\} = 3 \cdot \mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+4^2}\right\} + \frac{7}{2} \cdot \mathcal{L}^{-1}\left\{\frac{4}{(s-3)^2+4^2}\right\}$

$= 3 \cdot e^{3t} \cos 4t + \frac{7}{2} e^{3t} \sin 4t$.

Example 4 Use partial fractions to find the inverse Laplace transforms of the functions.

(1) $\dfrac{3s+19}{s^2+6s+34}$

(2) $\dfrac{s^2+1}{s^3-2s^2-8s}$

ANS: (1) $F(s) = \dfrac{3s+19}{s^2+6s+34} = \dfrac{3s+19}{s^2+6s+9-9+34} = \dfrac{3s+19}{(s+3)^2+25}$

$$= \frac{3(s+3)-9+19}{(s+3)^2+5^2} = 3\,\frac{s+3}{(s+3)^2+5^2} + 2\cdot\frac{5}{(s+3)^2+5^2}$$

Apply Table 1 with $a=-3$

$$\mathcal{L}^{-1}\{F(s)\} = 3\cdot\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+5^2}\right\} + 2\cdot\mathcal{L}^{-1}\left\{\frac{5}{(s+3)^2+5^2}\right\}$$

$$= 3\cdot e^{-3t}\cdot\cos 5t + 2\cdot e^{-3t}\sin 5t$$

(2) $R(s) = \dfrac{s^2+1}{s^3-2s^2-8s} = \dfrac{s^2+1}{s(s^2-2s-8)} = \dfrac{s^2+1}{s(s-4)(s+2)} = \dfrac{P(s)}{Q(s)}$

So $Q(s) = s(s+2)(s-4)$. By Rule 1, we can write

$$R(s) = \frac{s^2+1}{s(s+2)(s-4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-4}$$

$$= \frac{A(s+2)(s-4)+Bs(s-4)+Cs(s+2)}{s(s+2)(s-4)}$$

Compare the numerators, we have

$$s^2+1 = A(s+2)(s-4) + Bs(s-4) + Cs(s+2)$$

If we substitute $s=0$, $s=-2$, $s=4$, respectively, we have

$$\begin{cases} 1 = -8A \\ 5 = 12B \\ 17 = 24C \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{8} \\ B = \frac{5}{12} \\ C = \frac{17}{24} \end{cases}$$

Thus $R(s) = -\frac{1}{8}\cdot\frac{1}{s} + \frac{5}{12}\cdot\frac{1}{s+2} + \frac{17}{24}\cdot\frac{1}{s-4}$

$\mathcal{L}^{-1}\{R(s)\} = -\frac{1}{8} + \frac{5}{12}e^{-2t} + \frac{17}{24}e^{4t}$

Example 5 Use Laplace transforms to solve the initial value problem.

$$x'' - 4x = 3t; x(0) = x'(0) = 0.$$

ANS: Recall

$$\mathcal{L}\{x''\} = s^2 X(s) - s\cancel{x(0)}^{0} - \cancel{x'(0)}^{0}$$

$$\mathcal{L}\{x'' - 4x\} = \mathcal{L}\{3t\}$$ (Apply $\mathcal{L}$ on both sides of the eqn)

$$\Rightarrow s^2 X(s) - 4X(s) = \frac{3}{s^2}$$

$$\Rightarrow (s^2-4)X(s) = \frac{3}{s^2}$$

$$\Rightarrow X(s) = \frac{3}{s^2(s^2-4)}$$

$$= \frac{A}{s^2} + \frac{B}{s^2-4}$$

$$= \frac{A(s^2-4) + Bs^2}{s^2(s^2-4)}$$

$$= \frac{(A+B)s^2 - 4A}{s^2(s^2-4)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ -4A=3 \end{cases} \Rightarrow \begin{cases} A = -\frac{3}{4} \\ B = \frac{3}{4} \end{cases}$$

Thus $X(s) = -\frac{3}{4}\cdot\frac{1}{s^2} + \frac{3}{4}\cdot\frac{1}{s^2-4}$

$$\Rightarrow X(s) = \frac{3}{4}\left(\frac{1}{2}\frac{2}{s^2-2^2} - \frac{1}{s^2}\right)$$

$$\mathcal{L}^{-1}\{X(s)\} = \frac{3}{8}\sinh 2t - \frac{3}{4}t$$