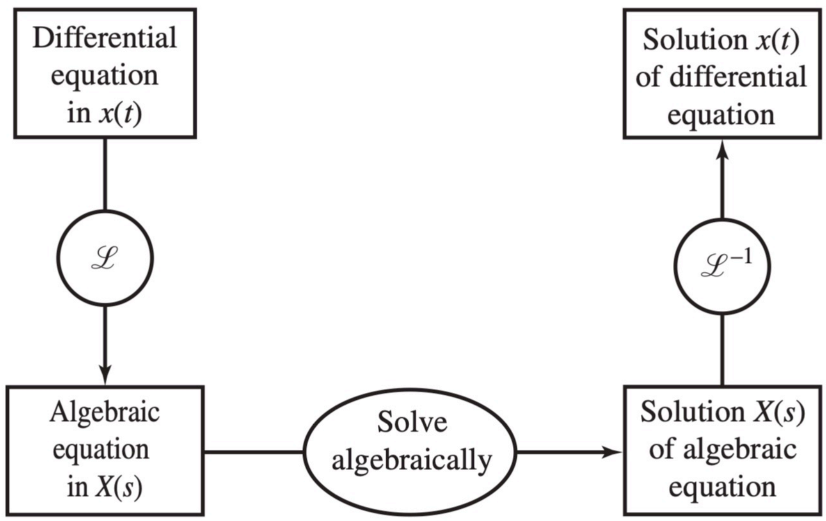
7.2 Transformation of Initial Value Problems

Recall in Section 7.1, we talked about the general process of using the Laplace transform to solve an initial value problem



Question How do we take Laplace transform to differential equation, say,

$$x'' + x = \sin 3t \ \ ?$$

From section 7.1, we know $\mathcal{L}\{\sin 3t\} = \dfrac{3}{s^2+9}$.

How to compute $\mathcal{L}\{x''(t)\}$ and $\mathcal{L}\{x'(t)\}$?

Theorem 1 Transforms of Derivatives

Suppose that the function $f(t)$ is continuous and piecewise smooth for $t = 0$ and is of exponential order as $t \to \infty$, so that there exist nonnegative constants M, c and T such that

$$|f(t)| \leq Me^{ct} \quad \text{for} \quad t \geq T. \tag{1}$$

Then $\mathcal{L}\{f'(t)\}$ exists for $s > c$, and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0). \tag{2}$$

Corollary. Transforms of Higher Derivatives

Suppose that the functions $f, f', f'', \cdots, f^{(n-1)}$ are continuous and piecewise smooth for $t \geq 0$, and that each of these functions satisfies the conditions in (1) with the same values of M and c. Then $\mathcal{L}\{f^{(n)}\}$ exists when $s > c$, and

$$\begin{aligned}\mathcal{L}\{f^{(n)}(t)\} &= s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-1} f'(0) - \cdots - f^{(n-1)}(0) \\ &= s^n F(s) - s^{n-1} f(0) - s^{n-1} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)\end{aligned} \tag{3}$$

Example: $\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0), \quad (n=2)$

$\mathcal{L}\{f^{(3)}(t)\} = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$

Example 1 Use Laplace transforms to solve the initial value problem.

$$x'' - x' - 6x = 0; \quad x(0) = 2, \quad x'(0) = -1. \tag{4}$$

ANS: Step 1. Take $\mathcal{L}$ on both sides of the equation. $\mathcal{L}\{x(t)\} = X(s)$

Step 2. Solve the equation for $X(s)$.

Step 3. Take $\mathcal{L}^{-1}$ on the solution of $X(s)$ to get $x(t)$.

Step 1. We compute

$$\mathcal{L}\{x'' - x' - 6x\} = \mathcal{L}\{0\} = 0$$

$$\Rightarrow \mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 6\mathcal{L}\{x\} = 0$$

$$\Rightarrow (s^2 X(s) - s x(0) - x'(0)) - (s X(s) - x(0)) - 6X(s) = 0$$

Step 2: $\Rightarrow s^2 X(s) - s \cdot 2 - (-1) - s X(s) + 2 - 6X(s) = 0$

$$\Rightarrow (s^2 - s - 6) X(s) - 2s + 3 = 0$$

$$\Rightarrow X(s) = \frac{2s-3}{s^2 - s - 6}$$

$$= \frac{2s-3}{(s-3)(s+2)} \quad \text{Assume} \quad \frac{A}{s-3} + \frac{B}{s+2}$$

compare

$$= \frac{A(s+2) + B(s-3)}{(s-3)(s+2)}$$

$$= \frac{(A+B)s + 2A - 3B}{(s-3)(s+2)}$$

$$\Rightarrow \begin{cases} A+B = 2 \\ 2A - 3B = -3 \end{cases} \Rightarrow \begin{cases} A = \frac{3}{5} \\ B = \frac{7}{5} \end{cases}$$

Then $X(s) = \frac{3}{5} \cdot \frac{1}{s-3} + \frac{7}{5} \cdot \frac{1}{s+2}$

Step 3. Take $\mathcal{L}^{-1}$ on both sides of the above equation. and recall $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{\underbrace{s+2}_{s-(-2)}}\right\}$$

$$\Rightarrow x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}$$

Example 2 Use Laplace transforms to solve the initial value problem.

$$x'' + x = \sin 3t; \quad x(0) = x'(0) = 0 \tag{5}$$

ANS: We take the Laplace transform on both sides of Eq (5)

$$\mathcal{L}\{x''\} + \mathcal{L}\{x\} = \mathcal{L}\{\sin 3t\}$$

$$\Rightarrow s^2 X(s) - s\cancel{x(0)}^{0} - \cancel{x'(0)}^{0} + X(s) = \frac{3}{s^2+9}$$

$$\Rightarrow (s^2+1) X(s) = \frac{3}{s^2+9}$$

$$\Rightarrow X(s) = \frac{3}{(s^2+1)(s^2+9)} \underset{\text{Assume}}{=} \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9}$$

Notice that there is no odd power of s on the left fraction. We can assume $A = C = 0$

Then we have

$$\frac{3}{(s^2+1)(s^2+9)} = \frac{B}{s^2+1} + \frac{D}{s^2+9}$$

$$= \frac{B(s^2+9) + D(s^2+1)}{(s^2+1)(s^2+9)}$$

$$= \frac{(B+D)s^2 + 9B + D}{(s^2+1)(s^2+9)}$$

$$\Rightarrow \begin{cases} B + D = 0 \\ 9B + D = 3 \end{cases} \Rightarrow \begin{cases} B = \frac{3}{8} \\ D = -\frac{3}{8} \end{cases}$$

Thus

$$X(s) = \frac{3}{8} \cdot \frac{1}{s^2+1} - \frac{3}{8} \frac{1}{s^2+9}$$

Recall $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

Then

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{3}{8} \cdot \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$x(t) = \frac{3}{8}\sin t - \frac{1}{8}\sin 3t$$

Ex.

Example 3 Use Laplace transforms to solve the initial value problem.

$$x'' + 4x' + 3x = 1; \quad x(0) = 0 = x'(0) \tag{6}$$

ANS: We have

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x'\} + 3\mathcal{L}\{x\} = \mathcal{L}\{1\}$$

$$\Rightarrow s^2 X(s) - s\cancel{x(0)}^{0} - \cancel{x'(0)}^{0} + 4\left[sX(s) - \cancel{x(0)}^{0}\right] + 3X(s) = \frac{1}{s}$$

$$\Rightarrow (s^2+4s+3)X(s) = \frac{1}{s}$$

$$\Rightarrow X(s) = \frac{1}{s(s^2+4s+3)} = \frac{1}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$= \frac{A(s+1)(s+3) + Bs(s+3) + Cs(s+1)}{s(s+1)(s+3)} = \frac{(A+B+C)s^2 + (4A+3B+C)\cdot s + 3A}{s(s+1)(s+3)}$$

$$\Rightarrow \begin{cases} 3A = 1 \\ 4A+3B+C = 0 \\ A+B+C = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{3} \\ B = -\frac{1}{2} \\ C = \frac{1}{6} \end{cases}$$

$$X(s) = \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{6}\cdot\frac{1}{s+3}$$

So

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$

Linear Systems

Example 4 Use Laplace transforms to solve the initial value problem.

$$\begin{aligned} x' &= 2x + y, \\ y' &= 6x + 3y, \\ x(0) &= 1,\ y(0) = -2 \end{aligned} \tag{7}$$

ANS: The transformed equations are

$$\begin{cases} \mathcal{L}\{x'\} = \mathcal{L}\{2x\} + \mathcal{L}\{y\} \\ \mathcal{L}\{y'\} = \mathcal{L}\{6x\} + \mathcal{L}\{3y\} \end{cases}$$

$$\Rightarrow \begin{cases} sX(s) - 1 = 2X(s) + Y(s) \\ sY(s) + 2 = 6X(s) + 3Y(s) \end{cases} \Rightarrow \begin{cases} (s-2)X(s) - Y(s) = 1 & \circledast \\ -6X(s) + (s-3)Y(s) = -2 & \circledast\circledast \end{cases}$$

We need to solve for $X(s)$ and $Y(s)$.

$\circledast \times (s-3) + \circledast\circledast$ gives

$$(s-3)(s-2)X(s) - 6X(s) = s-3-2$$

$$\Rightarrow \quad X(s) = \frac{s-5}{(s-3)(s-2)-6} = \frac{s-5}{s^2-5s} = \frac{s-5}{s(s-5)} = \frac{1}{s}$$

Then $\quad Y(s) = (s-2)X(s) - 1 = \frac{s-2}{s} - 1 = -\frac{2}{s}$

Thus $\quad \begin{cases} X(s) = \frac{1}{s} \\ Y(s) = -\frac{2}{s} \end{cases} \xrightarrow{\mathcal{L}^{-1}} \begin{cases} x(t) = \mathcal{L}^{-1}\{X(s)\} = 1 \\ y(t) = \mathcal{L}^{-1}\{Y(s)\} = -2 \end{cases}$

Additional Transform Techniques

Theorem 2. Transforms of Integrals

If $f(t)$ is a piecewise continuous function for $t \geq 0$ and satisfies the condition of exponential order $|f(t)| \leq Me^{ct}$ for $t \geq T$, then

$$\mathcal{L}\Big\{\int_0^t f(\tau)d\tau\Big\} = \frac{1}{s}\mathcal{L}\Big\{f(t)\Big\} = \frac{F(s)}{s} \tag{8}$$

for $s > c$. Equivalently,

$$\mathcal{L}^{-1}\Big\{\frac{F(s)}{s}\Big\} = \int_0^t f(\tau)d\tau \tag{9}$$

Example 5 Apply Theorem 2 to find the inverse Laplace transform of the function.

$$G(s) = \frac{1}{s^2(s-a)} = \frac{\frac{1}{s(s-a)}}{s} = \frac{\frac{\frac{1}{s-a}}{s}}{s} \tag{10}$$

$\mathcal{L}^{-1}\{e^{at}\}$

ANS:

ANS: We know $\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$

If we assume $F(s) = \frac{1}{s-a}$, then by eq (9) in Thm 2,

we have

$$\mathcal{L}^{-1}\Big\{\frac{F(s)}{s}\Big\} = \mathcal{L}^{-1}\Big\{\frac{1}{s(s-a)}\Big\} = \int_0^t \mathcal{L}^{-1}\Big\{\frac{1}{s-a}\Big\}d\tau$$

$$= \int_0^t e^{a\tau}d\tau = \frac{1}{a}\left[e^{a\tau}\right]\Big|_0^t = \frac{1}{a}(e^{at}-1)$$

Thus $\mathcal{L}^{-1}\Big\{\frac{1}{s(s-a)}\Big\} = \frac{1}{a}(e^{at}-1)$

We repeat the above technique to get

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\Big\{\frac{\overbrace{\frac{1}{s(s-a)}}^{F(s)}}{s}\Big\} \overset{\text{Eq (9)}}{=} \int_0^t \mathcal{L}^{-1}\Big\{\frac{1}{s(s-a)}\Big\}d\tau$$

$$= \int_0^t \frac{1}{a}(e^{a\tau}-1)\,d\tau$$

$$= \frac{1}{a}\left[\frac{1}{a}e^{a\tau} - \tau\right]\Big|_0^t = \frac{1}{a}\left[\frac{1}{a}e^{at} - t - \frac{1}{a}\right]$$

$$= \frac{1}{a^2}\left[e^{at} - at - 1\right]$$

Example 6 Apply Theorem 2 to find the inverse Laplace transforms of the function.

$$F(s) = \frac{1}{s^2(s^2+1)} \tag{11}$$

ANS: We know $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

Then by Thm 2

$$\mathcal{L}^{-1}\left\{\frac{\frac{1}{s^2+1}}{s}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} d\tau = \int_0^t \sin\tau \, d\tau = -\cos\tau \Big|_0^t$$

$$= 1 - \cos t$$

Thus $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$

We apply Thm 2 again.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{s(s^2+1)}}{s}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} d\tau$$

$$= \int_0^t (1-\cos\tau)\, d\tau = \left[\tau - \sin\tau\right]\Big|_0^t = t - \sin t$$

Thus $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t - \sin t$