Chapter 7 Laplace Transform Methods

Introduction

Recall in Chapter 3; we talked about the method of solving the mass-spring-dashpot system

$$mx'' + cx' + kx = F(t)$$

In practice, the forcing term $F(t)$ has discontinuities. In this case, the Laplace transform method is a prefered method to solve Eq (1).

The Laplace transformation $\mathcal{L}$ can be viewed as an analogy of the differentiation operator D:

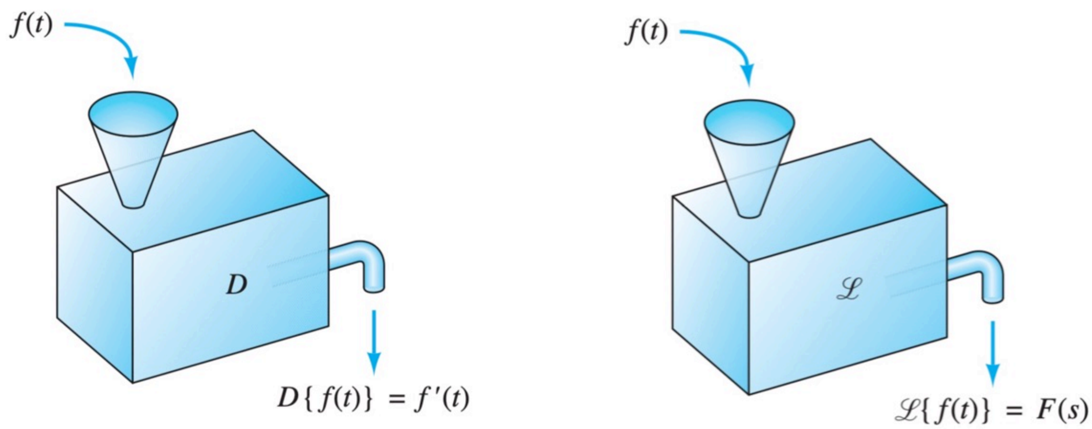


Figure: Transformation of a function: $\mathcal{L}$ in analogy with D

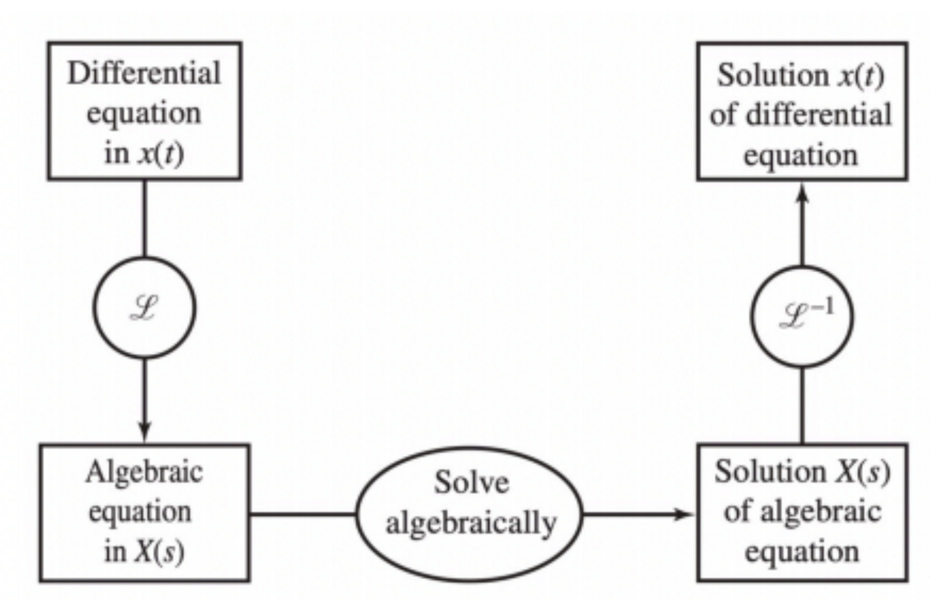


Figure. Using the Laplace transform to solve an initial value problem

7.1 Laplace Transforms and Inverse Transforms

Definition The Laplace Transform

Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is the function F defined as follows:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

for all values of s for which the improper integral converges.

Recall that an improper integral over an infinite interval is defined as a limit of integrals over bounded intervals:

$$\int_a^\infty g(t)dt = \lim_{b \to \infty} \int_a^b g(t)dt.$$

If the limit (2) exists, then we say that the improper integral **converges**; otherwise, it **diverges** or fails to exist.

Example 1 Apply the definition in (2) to find the Laplace transform of the given function.

$$f(t) = e^{3t+1}$$

ANS: By the def. above, the Laplace transform of $f(t)$ is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot e^{3t+1}\, dt$$

$$= \int_0^\infty e^{-st} \cdot e^{3t} \cdot e\, dt$$

$$= e \int_0^\infty e^{-(s-3)t}\, dt$$

We compute $\int e^{-(s-3)t}\, dt$

Let $u = -(s-3)t$, then $du = (3-s)\,dt$

Thus $dt = \frac{1}{3-s}\, du$

So

$$\int e^{-(s-3)t}\,dt = \int e^{u}\frac{1}{3-s}\,du = \frac{1}{3-s}\int e^{u}\,du$$

$$= \frac{1}{3-s}e^{u} = \frac{1}{3-s}e^{-(s-3)t}$$

Then $F(s) = \dfrac{e}{3-s}\Big[e^{-(s-3)t}\Big]_0^{\infty}$

$$= \frac{e}{3-s}\lim_{b\to\infty}\Big[e^{-(s-3)b} - \overset{1}{\cancel{e^{-(s-3)\cdot 0}}}\Big]$$

If $s-3>0$, then $e^{-(s-3)b} \to 0$ as $b\to\infty$

So $F(s) = \dfrac{e}{3-s}(0-1) = \dfrac{e}{s-3}$ for $s>3$

Note $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ $(s>a)$

$$\mathcal{L}\{e^{3t+1}\} = \mathcal{L}\{e^{3t}\cdot e\} = e\,\mathcal{L}\{e^{3t}\}$$

$$= e\cdot\frac{1}{s-3} \quad (s>3)$$

Reading Material: Gamma function $\Gamma(x)$ and $\mathcal{L}\{t^a\}$

The Laplace transform $\mathcal{L}\{t^a\}$ of a power function is most conveniently expressed in terms of the **gamma function** $\Gamma(x)$, which is defined for $x > 0$ by the formula

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

- $\Gamma(1) = 1$
- $\Gamma(x+1) = x\Gamma(x)$
- $\Gamma(n+1) = n!$

Example 2 Apply the definition in (2) to find the Laplace transform of the given function.

$$f(t) = t^a, \qquad \text{where } a \text{ real and } a > -1.$$

ANS: By def.

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st} t^a \, dt$$

Let $u = st$, then $t = \frac{u}{s}$

$du = s\,dt$, then $dt = \frac{du}{s}$

Substitute these into the integral, we have

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st} t^a dt = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \cdot \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^{(a+1)-1} du$$

$$= \frac{1}{s^{a+1}} \Gamma(a+1)$$

Thus $\mathcal{L}\{t^a\} = \dfrac{\Gamma(a+1)}{s^{a+1}}$ for all $a > -1$.

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

If a is an integer, we have

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \quad (s > 0)$$

For example, $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\{t^2\} = \frac{2}{s^3}$, $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$, ...

Note: $\int e^{bx} \sin ax \quad dx = \frac{1}{a^2+b^2} e^{bx} (b \sin ax - a \cos ax)$

Example 3 Apply the definition in (2) to find the Laplace transform of the given function.

$$f(t) = \cos kt$$

ANS: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot \cos kt \, dt$

Check the table of integrals.

$$\int e^{bx} \cos ax \quad dx = \frac{1}{a^2+b^2} e^{bx} (a \sin ax + b \cos ax)$$

Then $\int e^{-st} \cdot \cos kt \, dt = \frac{1}{s^2+k^2} e^{-st} (k \sin kt - s \cos kt)$

Then $\int_0^\infty e^{-st} \cos kt \, dt = \lim_{b \to \infty} \left[\frac{e^{-st}(k \sin kt - s \cos kt)}{s^2+k^2} \right] \Big|_0^b$

So $\mathcal{L}\{\cos kt\} = 0 - \frac{-s}{s^2+k^2} = \boxed{\frac{s}{s^2+k^2} \quad (s>0)}$

Another Method

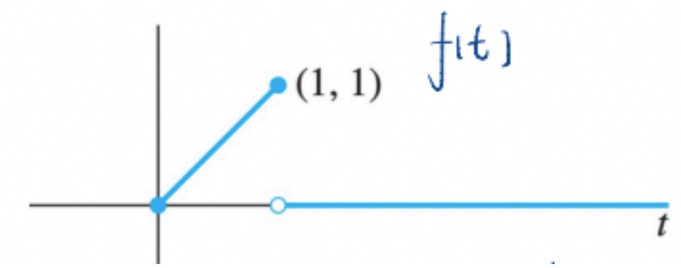
Recall $\left.\begin{aligned} e^{ikt} &= \cos kt + i \sin kt \\ e^{-ikt} &= \cos kt - i \sin kt \end{aligned}\right\}$. then

$\cos kt = \frac{e^{ikt} + e^{-ikt}}{2}$, $\sin kt = \frac{e^{ikt} - e^{-ikt}}{2i}$.

$\mathcal{L}\{\cos kt\} = \mathcal{L}\left\{ \frac{e^{ikt} + e^{-ikt}}{2} \right\} = \frac{1}{2} \left(\mathcal{L}\{e^{ikt}\} + \mathcal{L}\{e^{-ikt}\} \right)$

$= \frac{1}{2} \left(\frac{1}{s-ik} + \frac{1}{s+ik} \right) = \frac{1}{2} \left(\frac{s+ik+s-ik}{s^2+k^2} \right) = \frac{s}{s^2+k^2}$

Example 4 Apply the definition in (2) to find directly the Laplace transform of the given function described by the graph.



ANS: From the graph,

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

Thus

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt = \int_0^1 e^{-st} \cdot t\, dt + \int_1^\infty \cancelto{0}{e^{-st} \cdot 0}\, dt$$

$$= \int_0^1 e^{-st} \cdot t\, dt.$$

Note $\int t e^{-st} dt$

$\int u\,dv = uv - \int v\,du$

$$= -\frac{1}{s} \int t\, d e^{-st}$$

$$= -\frac{1}{s}\left[t e^{-st} - \int e^{-st} dt\right]$$

$$= -\frac{1}{s}\left[t e^{-st} - \frac{1}{-s}\int e^{-st} d(-st)\right]$$

$$= -\frac{1}{s}\left[t e^{-st} + \frac{1}{s} e^{-st}\right] = -\frac{(st+1)e^{-st}}{s^2}$$

Thus

$$\mathcal{L}\{f(t)\}$$

$$= \left[-\frac{(st+1)e^{-st}}{s^2}\right]\Bigg|_0^1$$

$$= -\frac{(s+1)e^{-s}}{s^2} - \left(-\frac{\cancelto{1}{e^0}}{s^2}\right)$$

$$= \boxed{\frac{1-(s+1)e^{-s}}{s^2}}$$

Linearity of Transforms

Theorem 1. Linearity of the Laplace Transform

If a and b are constants, then

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

for all s such that the Laplace transforms of the functions f and g both exists.

Example 5 Use the transforms in Fig. 7.1.2 to find the Laplace transforms of the functions of the following problems.

(1) $f(t) = \sin 3t + \cos 3t$

(2) $f(t) = (1+t)^2$

$f(t)$	$F(s)$	
1	$\dfrac{1}{s}$	$(s>0)$
t	$\dfrac{1}{s^2}$	$(s>0)$
$t^n \quad (n \geqq 0)$	$\dfrac{n!}{s^{n+1}}$	$(s>0)$
$t^a \quad (a > -1)$	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	$(s>0)$
e^{at}	$\dfrac{1}{s-a}$	$(s>a)$
$\cos kt$	$\dfrac{s}{s^2+k^2}$	$(s>0)$
$\sin kt$	$\dfrac{k}{s^2+k^2}$	$(s>0)$
$\cosh kt$	$\dfrac{s}{s^2-k^2}$	$(s>\lvert k\rvert)$
$\sinh kt$	$\dfrac{k}{s^2-k^2}$	$(s>\lvert k\rvert)$
$u(t-a)$	$\dfrac{e^{-as}}{s}$	$(s>0)$

FIGURE 7.1.2. A short table of Laplace transforms.

ANS:

(1) $\mathcal{L}\{f(t)\}$

$= \mathcal{L}\{\sin 3t + \cos 3t\}$

$= \mathcal{L}\{\sin 3t\} + \mathcal{L}\{\cos 3t\}$

$= \dfrac{3}{s^2+9} + \dfrac{s}{s^2+9}$

$= \dfrac{3+s}{s^2+9} \quad (s>0)$

(2) $\mathcal{L}\{(1+t)^2\}$

$= \mathcal{L}\{1 + 2t + t^2\}$

$= \mathcal{L}\{1\} + 2\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$

$= \dfrac{1}{s} + 2 \cdot \dfrac{1}{s^2} + \dfrac{2!}{s^{2+1}}$

$= \dfrac{1}{s} + \dfrac{2}{s^2} + \dfrac{2}{s^3} \quad (s>0)$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$ $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Example 6 Use the transforms in Fig. 7.1.2 to find the Laplace transforms of the functions of the following problem.

$$f(t) = \cos^2 3t$$

ANS: Note $\cos^2 3t = \frac{1}{2}(1+\cos 6t)$

$$\mathcal{L}\{\cos^2 3t\} = \mathcal{L}\{\tfrac{1}{2}(1+\cos 6t)\} = \tfrac{1}{2}\left(\mathcal{L}\{1\} + \mathcal{L}\{\cos 6t\}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+36}\right)$$

Inverse Transforms

Definition Inverse Transforms

If $F(s) = \mathcal{L}\{f(t)\}$, then we call $f(t)$ the inverse Laplace transform of $F(s)$ and write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Example

Since $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$.

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$.

Example 7 Use the transforms in Fig. 7.1.2 to find the inverse Laplace transforms of the given functions.

(i) $F(s) = \dfrac{3}{s+5}$

(ii) $F(s) = \dfrac{3}{s^4}$

(iii) $F(s) = \dfrac{3s+1}{s^2+4}$

$f(t)$	$F(s)$	
1	$\dfrac{1}{s}$	$(s > 0)$
t	$\dfrac{1}{s^2}$	$(s > 0)$
$t^n \quad (n \geqq 0)$	$\dfrac{n!}{s^{n+1}}$	$(s > 0)$
$t^a \quad (a > -1)$	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	$(s > 0)$
e^{at}	$\dfrac{1}{s-a}$	$(s > a)$
$\cos kt$	$\dfrac{s}{s^2+k^2}$	$(s > 0)$
$\sin kt$	$\dfrac{k}{s^2+k^2}$	$(s > 0)$
$\cosh kt$	$\dfrac{s}{s^2-k^2}$	$(s > \lvert k \rvert)$
$\sinh kt$	$\dfrac{k}{s^2-k^2}$	$(s > \lvert k \rvert)$
$u(t-a)$	$\dfrac{e^{-as}}{s}$	$(s > 0)$

FIGURE 7.1.2. A short table of Laplace transforms.

ANS:

(1) $\mathcal{L}^{-1}\left\{\dfrac{3}{s+5}\right\}$

$= 3\,\mathcal{L}^{-1}\left\{\dfrac{1}{s+5}\right\}$

$= 3\,\mathcal{L}^{-1}\left\{\dfrac{1}{s-(-5)}\right\}$

$= 3e^{-5t}$

(2). $\mathcal{L}^{-1}\left\{\dfrac{3}{s^4}\right\}$

$= 3\,\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\}$

$= 3\,\mathcal{L}^{-1}\left\{\dfrac{3!}{s^{3+1}} \cdot \dfrac{1}{3!}\right\}$

$= \dfrac{3}{3!}\,\mathcal{L}^{-1}\left\{\dfrac{3!}{s^{3+1}}\right\}$

$= \dfrac{1}{2}t^3$

(3) $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{s^2+4}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{3s}{s^2+2^2}\right\} + \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+2^2}\right\}$

$= 3\cdot\cos 2t + \dfrac{1}{2}\sin 2t.$