5.7 Nonhomogeneous Linear Systems

Given the nonhomogeneous first-order linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$$

where A is an $n \times n$ constant matrix and the "nonhomogeneous term" $\mathbf{f}(t)$ is a given continuous vector-valued function.

A general solution of Eq (1) has the form

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t),$$

where

- $\mathbf{x}_c = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$ is a general solution of the associated homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$,
- $\mathbf{x}_p(t)$ is a single particular solution of the original nonhomogeneous system in (1).

Undetermined Coefficients

Example 1 Apply the method of undetermined coefficients to find a particular solution of the following system.

$$\begin{cases} x' = x + 2y + 3 \\ y' = 2x + y - 2 \end{cases}$$

ANS: We assume $\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ for some numbers a, b.

Then we substitute x_p, y_p into the eqns.

$$\begin{cases} 0 = a + 2b + 3 \\ 0 = 2a + b - 2 \end{cases} \Rightarrow \begin{cases} a + 2b = -3 \Rightarrow 2a + 4b = -6 \\ 2a + b = 2 \end{cases}$$

$$\Rightarrow 3b = -8 \quad \Rightarrow \quad b = -\frac{8}{3}$$

$$a = -3 - 2b = -3 + 2 \cdot \frac{8}{3} = -3 + \frac{16}{3} = \frac{7}{3}$$

Thus $\begin{cases} x_p = \frac{7}{3} \\ y_p = -\frac{8}{3} \end{cases}$

Recall that if we want to find $x_p(t)$ for the equation $x'' - x = e^t$, we assume $x_p = a\,te^t$ since e^t is a solution for the homogeneous equation $x'' - x = 0$.

Similarly, in general cases, we need to check the solution for $\mathbf{x}_c$ for the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

For example,

Example 2 Apply the method of undetermined coefficients to find a particular solution of the following system.

$$\vec{x}' = A\vec{x} + \vec{f}(t)$$

$$x' = 2x + y + 2e^t$$
$$y' = x + 2y - 3e^t$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^t$$

$$\cancel{\vec{x}_p(t) = \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} e^t}$$

ANS: We first consider the homogeneous part

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) = 0$$

$\lambda = 1$ or $\lambda = 3$.

So $\vec{v}_1 e^t$ appears in the solution for the homogeneous eqn

So we assume.

$$\vec{x}_p(t) = \underset{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}{\underset{\uparrow}{\vec{a}}}\, e^t + \underset{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}{\underset{\uparrow}{\vec{b}}}\, te^t$$

$$\Rightarrow \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^t \Rightarrow \begin{bmatrix} x_p' \\ y_p' \end{bmatrix} = \begin{bmatrix} (a_1+b_1)e^t + b_1 te^t \\ (a_2+b_2)e^t + b_2 te^t \end{bmatrix}$$

We substitute x_p, y_p, x_p', y_p' into the system.

we get

$$\begin{cases} (a_1+b_1)e^t + b_1 te^t = (2a_1+a_2)e^t + (2b_1+b_2)te^t + 2e^t \\ (a_2+b_2)e^t + b_2 te^t = (a_1+2a_2)e^t + (b_1+2b_2)te^t - 3e^t \end{cases}$$

Compare the coefficients for e^t, te^t. we have

$$\begin{cases} a_1+b_1 -2a_1 - a_2 - 2 = 0 & \Rightarrow -a_1+b_1-a_2-2=0 \\ b_1-2b_1-b_2=0 & \Rightarrow -b_1-b_2=0 \\ a_2+b_2-a_1-2a_2+3=0 & \Rightarrow -a_2+b_2-a_1+3=0 \\ b_2-b_1-2b_2=0 & \Rightarrow -b_1-b_2=0 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = \frac{1}{2} \\ a_2 = 0 \\ b_1 = \frac{5}{2} \\ b_2 = -\frac{5}{2} \end{cases}$$

Then $\vec{x}_p = ae^t + bte^t$

$$\Rightarrow \begin{cases} x_p = \frac{1}{2}e^t + \frac{5}{2}te^t \\ y_p = -\frac{5}{2}te^t \end{cases}$$

Example 3 Apply the method of undetermined coefficients to find a particular solution of the following system.

$$\vec{x}' = A\vec{x} + \vec{f}(t)$$

$$x' = x - 5y + \cos 2t$$
$$y' = x - y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos 2t \\ 0 \end{bmatrix}$$

$$\cancel{\vec{x}_p = \vec{a}\sin 2t + \vec{b}\cos 2t}$$

ANS: We first look at the corresponding homogeneous eqn.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

This means $\sin 2t$ & $\cos 2t$ appears in the solution for the homogeneous part

Use the method in §5.2 we find

$$\vec{x}_c = c_1 \begin{bmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} 2\cos 2t + \sin 2t \\ \sin 2t \end{bmatrix}$$

We assume

$$\vec{x}_p = \begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix} = \underset{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}{\vec{a}} \sin 2t + \underset{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}{\vec{b}} \cos 2t + \underset{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}{\vec{c}}\, t \sin 2t + \underset{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}}{\vec{d}}\, t \cos 2t$$

The computation is quite long, so we list some of the result to save some time in our class. We have

$$x_p' = (2a_1 + d_1)\cos(2t) + 2c_1 t\cos(2t) + (-2b_1 + c_1)\sin(2t) - 2d_1 t\sin(2t)$$

$$y_p' = (2a_2 + d_2)\cos(2t) + (-2b_2 + c_2)\sin(2t) + 2c_2 t\cos(2t) - 2d_2 t\sin(2t)$$

Compare the coefficients for $\sin(2t)$, $\cos(2t)$, $t\sin(2t)$ and $t\cos(2t)$ in the equations $x_p' = x_p - 5y_p + \cos 2t$ and $y_p' = x_p - y_p$, we have the following 8 equations:

$$
\begin{cases}
2c_1 - d_1 + 5d_2 = 0 \\
-1 + 2a_1 - b_1 + 5b_2 + d_1 = 0 \\
-a1 + 5a_2 - 2b_1 + c_1 = 0 \\
-c_1 + 5c_2 - 2d_1 = 0 \\
2a_2 - b_1 + b_2 + d_2 = 0 \\
2c_2 - d_1 + d_2 = 0 \\
-a_1 + a_2 - 2b_2 + c_2 = 0 \\
-c_1 + c_2 - 2d_2 = 0
\end{cases}
$$

We use calculator or computer to solve for these 8 variables, we find a solution

$a_1 = \frac{1}{4}$, $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{4}$, $d_1 = \frac{1}{2}$ and $a_2 = b_2 = d_2 = 0$.

Thus $x_p(t) = \frac{1}{4}(\sin 2t + 2t\cos 2t + t\sin 2t)$ and $y_p(t) = \frac{1}{4}t\sin 2t$.

Using the **Variation of Parameters** discussed later in this section, we can use **Matlab** code to solve for this question. In Matlab, we writeUsing the **Variation of Parameters** discussed later in this section, we can use **Matlab** code to solve for this question. In **Matlab**, we write

```
% We first define variable s and t
syms t s
% We define the initial condition as follows
x0 = [0;0];
% The matrix A is given in the question
A = [1 -5; 1 -1];
% We define a column vector fs as the column vector function f(t)
% Note here we replace t with s
fs = [cos(2.*s);0];
% We compute the definite integral in Eq(12) in our notes
integral = int(expm(-A.*s)*fs, 0, t);
% We compute x(t) using Eq(12)
solutiontoexample3 = simplify(expm(A.*t)*(x0 + integral))
```

After running the code we have the output

```
solution
 [1/4(t+1) sin (2 t)+ 1/2 cos (2 t) t]
 [                                   ]
 [              1/4 sin (2 t) t      ]

```

Variation of Parameters

We want to find a particular solution $\mathbf{x}_p$ of the nonhomogeneous linear system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t),$$

given that we have already found a general solution

$$\mathbf{x}_c = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

of the associated homogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}.$$

THEOREM 1 Variation of Parameters

If $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on some interval where $\mathbf{P}(t)$ and $\mathbf{f}(t)$ are continuous, then a particular solution of the non- homogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t),$$

is given by

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)dt.$$

If $\mathbf{P}(t) \equiv \mathbf{A}$, we can use $\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$.

Then

$$\boxed{\mathbf{x}_p(t) = e^{\mathbf{A}t}\int e^{-\mathbf{A}s}\mathbf{f}(s)ds}$$

Consider the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0$$

Then the solution is given by

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}(s)}\mathbf{f}(s)ds}$$

Note there were typos in our textbook for Eq(10) and Eq(12) on page 367.

Example 3 Given that $X_1(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $X_2(t) = \begin{bmatrix} 2t+1 \\ t \end{bmatrix}$ are a pair of linearly independent solutions to the homogenous system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Use the method of variation of parameters to solve the initial value problem

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 36t^2 \\ 6t \end{bmatrix}}_{\vec{f}(t)}; \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{x}_0$$

ANS: We use the formula.

$$\vec{x}(t) = e^{At}\underbrace{\cancel{\vec{x}_0}}_{\vec{0}} + e^{At}\int_0^t e^{-A(s)} f(s)\,ds$$

Since $\vec{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$\vec{x}(t) = e^{At}\int_0^t e^{-A(s)} f(s)\,ds$$

Recall $e^{At} = \Phi(t)\Phi^{-1}(0)$

$$\Phi(t) = [X_1 \quad X_2] = \begin{bmatrix} 2 & 2t+1 \\ 1 & t \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \Phi^{-1}(0) = \frac{1}{2\cdot 0 - 1\times 1}\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

Thus $$e^{At} = \begin{bmatrix} 2 & 2t+1 \\ 1 & t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2t+1 & -4t \\ t & 1-2t \end{bmatrix}$$

Then

$$\vec{x}(t) = e^{At}\int_0^t e^{-As} f(s)\,ds$$

$$= \begin{bmatrix} 2t+1 & -4t \\ t & 1-2t \end{bmatrix} \boxed{\int_0^t \begin{bmatrix} 2(-s)+1 & -4(-s) \\ -s & 1-2(-s) \end{bmatrix} \begin{bmatrix} 36s^2 \\ 6s \end{bmatrix} ds}$$

$$= 6\int_0^t \begin{bmatrix} -2s+1 & 4s \\ -s & 1+2s \end{bmatrix} \begin{bmatrix} 6s^2 \\ s \end{bmatrix} ds$$

$$= 6\int_0^t \begin{bmatrix} -12s^3+10s^2 \\ -6s^3+2s^2+s \end{bmatrix} ds$$

$$= 6\begin{bmatrix} \left[-\frac{12}{4}s^4+\frac{10}{3}s^3\right]\Big|_0^t \\ \left[-\frac{6}{4}s^4+\frac{2}{3}s^3+\frac{1}{2}s^2\right]\Big|_0^t \end{bmatrix}$$

$$= \begin{bmatrix} -18t^4+20t^3 \\ -9t^4+4t^3+3t^2 \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} 2t+1 & -4t \\ t & 1-2t \end{bmatrix} \begin{bmatrix} -18t^4+20t^3 \\ -9t^4+4t^3+3t^2 \end{bmatrix}$$

$$\Rightarrow \vec{x}(t) = \begin{bmatrix} 8t^3 + 6t^4 \\ 3t^2 - 2t^3 + 3t^4 \end{bmatrix}$$

Exercise 4 Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} te^{-2t}, \quad \mathbf{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

The solutions are on page 368 of the textbook.

Example 5 In the following question, use the method of variation of parameters (and perhaps a computer algebra system) to solve the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(a) = \mathbf{x}_a.$$

where $\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 18e^{2t} \\ 30e^{2t} \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $e^{\mathbf{A}t} = \dfrac{1}{4}\begin{bmatrix} -e^{-t} + 5e^{3t} & e^{-t} - e^{3t} \\ -5e^{-t} + 5e^{3t} & 5e^{-t} - e^{3t} \end{bmatrix}$.

Ans: To solve this by hand, we need to follow the steps discussed in **Example 3**. We provide a computer code to solve this question.

In **Matlab**, we write the following code:

```
% We first define variable s and t
syms t s
% We define the initial condition as follows
x0 = [0;0];
% The matrix A is given in the question
A = [4 -1; 5 -2];
% We define a column vector fs as the column vector function f(t)
% Note here we replace t with s
fs = [18*exp(2*s); 30*exp(2*s)];
% We compute the definite integral in Eq(12) in our notes
integral = int(expm(-A.*s)*fs, 0, t);
% We compute x(t) using Eq(12)
solutiontoexample4 = simplify(expm(A.*t)*(x0 + integral))
```

After running the code, we get the following result

```
solution =

                [15  exp (3 t)- 14 exp (2 t)  -   exp (-t)]
                [                                         ]
                [15  exp (3 t) - 10 exp (2 t) - 5 exp (-t)]

```

Thus

$x_1(t) = e^{-t}\left(e^{3t}\left(15e^t - 14\right) - 1\right)$

$x_2(t) = e^{-t}\left(5e^{3t}\left(3e^t - 2\right) - 5\right)$