5.5 Multiple Eigenvalue Solutions

In this section we discuss the situation when the characteristic equation

$$|\mathbf{A}-\lambda\mathbf{I}|=0 \tag{1}$$

does not have n distinct roots, and thus has at least one repeated root.

An eigenvalue is of **multiplicity** k if it is a k-fold root of Eq. (1).

1. Complete Eigenvalues

- We call an eigenvalue of multiplicity k **complete** if it has k linearly independent associated eigenvectors.
- If every eigenvalue of the matrix $\mathbf{A}$ is complete, then - because eigenvectors associated with different eigenvalues are linearly independent-it follows that $\mathbf{A}$ does have a complete set of n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (each repeated with its multiplicity).
- In this case a general solution of Eq. (1) is still given by the usual combination

$$\mathbf{x}(t)=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}+\cdots+c_n\mathbf{v}_ne^{\lambda_n t}$$

Example 1 (An example of a complete eigenvalue)

Find the general solution of the systems in the following problem.

$$\mathbf{x}'=\begin{bmatrix}2&0&0\\-7&9&7\\0&0&2\end{bmatrix}\mathbf{x} \tag{1}$$

ANS: The characteristic equation of the coefficient matrix A is

$$|A-\lambda I|=\begin{vmatrix}2-\lambda&0&0\\-7&9-\lambda&7\\0&0&2-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}9-\lambda&7\\0&2-\lambda\end{vmatrix}=(2-\lambda)^2(9-\lambda)=0$$

Thus $\lambda=2,2,9$.

- Case $\lambda_1=2$. We solve $(A-\lambda_1 I)\mathbf{v}=\mathbf{0}$.

  That is, $(A-\lambda_1 I)\vec{v}=\begin{bmatrix}0&0&0\\-7&7&7\\0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

  $\Rightarrow -a+b+c=0$

  - If $c=0, -a+b=0$.

We can take $a = b = 1$. Then $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector to $\lambda_1 = 2$.

- If $b = 0$, then $-a + c = 0$.

  We can take $a = c = 1$. Then $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is another eigenvector to $\lambda_1 = 2$.

  Note $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

- Case $\lambda_2 = 9$. We solve

$$(A - 9I)\mathbf{v}_3 = \begin{bmatrix} -7 & 0 & 0 \\ -7 & 0 & 7 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$\Rightarrow \begin{cases} a = 0 \\ a + c = 0 \\ c = 0 \end{cases}.$$

Let $b = 1$. Then $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponds to $\lambda_2 = 9$.

Then the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{4t} \tag{3}$$

2. Defective Eigenvalues

(i.e. λ has less than k. linearly independent eigenvectors)

- An eigenvalue λ of multiplicity $k > 1$ is called **defective** if it is not complete.
- If the eigenvalues of the $n \times n$ matrix $\mathbf{A}$ are not all complete, then the eigenvalue method will produce fewer than the needed n linearly independent solutions of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- An example of this is the following **Example 2**.
- The defective eigenvalue $\lambda_1 = 5$ in Example 2 has multiplicity $k = 2$, but it has only 1 associated eigenvector.

The Case of Multiplicity $k = 2$

Remark: The method of finding the solutions is summarized in the **Algorithm Defective Multiplicity 2 Eigenvalues**. The following steps explain why this algorithm works.

- Let us consider the case $k = 2$, and suppose that we have found (as in Example 2) that there is only a single eigenvector $\mathbf{v}_1$ associated with the defective eigenvalue λ.
- Then at this point we have found only the single solution $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

> Recall when solving $ax'' + bx' + cx = 0$.
>
> If $ar^2 + br + c = 0$ has repeated roots. Then two linearly independent solutions are e^{rt}, te^{rt}

- By analogy with the case of a repeated characteristic root for a single linear differential equation, we might hope to find a second solution of the form

$$\mathbf{x}_2(t) = (\mathbf{v}_2 t)e^{\lambda t} = \mathbf{v}_2 t e^{\lambda t}$$

- When we substitute $\mathbf{x} = \mathbf{v}_2 t e^{\lambda t}$ in $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we get the equation

$$\mathbf{v}_2 e^{\lambda t} + \lambda \mathbf{v}_2 t e^{\lambda t} = \mathbf{A}\mathbf{v}_2 t e^{\lambda t}$$

- But because the coefficients of both $e^{\lambda t}$ and $te^{\lambda t}$ must balance, it follows that $\mathbf{v}_2 = \mathbf{0}$, and hence that $\mathbf{x}_2(t) \equiv \mathbf{0}$.
- This means that - contrary to our hope - the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ does not have a nontrivial solution of the form we assumed.
- Let us extend our idea slightly and replace $\mathbf{v}_2 t$ with $\mathbf{v}_1 t + \mathbf{v}_2$.
- Thus we explore the possibility of a second solution of the form

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero constant vectors.

- When we substitute $\mathbf{x} = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$ in $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we get the equation

$$\mathbf{v}_1 e^{\lambda t} + \lambda \mathbf{v}_1 t e^{\lambda t} + \lambda \mathbf{v}_2 e^{\lambda t} = \mathbf{A}\mathbf{v}_1 t e^{\lambda t} + \mathbf{A}\mathbf{v}_2 e^{\lambda t}$$

$$\Leftrightarrow \begin{cases} \lambda \vec{v}_1 \cancel{te^{\lambda t}} = A\vec{v}_1 \cancel{te^{\lambda t}} \\ \vec{v}_1 \cancel{e^{\lambda t}} + \lambda \vec{v}_2 \cancel{e^{\lambda t}} = A v_2 \cancel{e^{\lambda t}} \end{cases} \Leftrightarrow \begin{cases} \lambda \vec{v}_1 = A\vec{v}_1 \\ \vec{v}_1 + \lambda \vec{v}_2 = A\vec{v}_2 \end{cases} \Leftrightarrow \begin{cases} (A-\lambda I)\vec{v}_1 = \vec{0} \\ (A-\lambda I)\vec{v}_2 = \vec{v}_1 \end{cases}$$

- We equate coefficients of $e^{\lambda t}$ and $te^{\lambda t}$ here, and thereby obtain the two equations

$$\boxed{(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_1 = \mathbf{0}} \quad \text{and} \quad \boxed{(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1}$$

that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ must satisfy in order for

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$$

to give a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

- Note that the first of these two equations merely confirms that $\mathbf{v}_1$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue λ.
- Then the second equation says that the vector $\mathbf{v}_2$ satisfies

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\left[(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2\right] = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}$$

- It follows that, in order to solve the two equations simultaneously, it suffices to find a solution $\mathbf{v}_2$ of the single equation $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$ such that the resulting vector $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2$ is nonzero.

Algorithm Defective Multiplicity 2 Eigenvalues

1. First find nonzero solution $\mathbf{v}_2$ of the equation

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0} \tag{4}$$

such that

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 \tag{5}$$

is nonzero, and therefore is an eigenvector $\mathbf{v}_1$ associated with λ.

2. Then form the two independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t} \tag{6}$$

and

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} \tag{7}$$

of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to λ.

Rmk: ① By Thm 3 in §5.1, we need to find $\vec{x}_1(t)$ and $\vec{x}_2(t)$ that are linearly independent.

② Note the above algorithm produces two solutions $\vec{x}_1$, $\vec{x}_2$ that linearly independent.

③ Note $\vec{v}_1$, $\vec{v}_2$ are not unique!

But they satisfy $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

Example 2 (λ with multiplicity 2 , and λ is defective)

Find the general solution of the system in the following problem. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the system.

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \mathbf{x} \tag{8}$$

ANS: Find the eigenvalues of A

$$0 = |A-\lambda I| = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = (1-\lambda)(9-\lambda)+16 = \lambda^2-10\lambda+25$$
$$= (\lambda-5)^2 = 0$$

$\Rightarrow \lambda = 5$ with multiplicity 2.

$$(A-5I)\vec{v} = \vec{0} = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow a+b=0 \Rightarrow a=-b$$

The eigenvector corresponds to $\lambda=5$ is a multiple of $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Thus λ has multiplicity 2 but only has one linearly independent eigenvector.

We apply the above algorithm to find $\vec{v}_2$ and $\vec{v}_1$.

· We solve

$$\vec{0} = (A-5I)^2 \vec{v}_2 = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So any a, b satisfy this equation.

Let's choose $a=1$, $b=0$. Then $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

· We compute

$$(A-5I)\vec{v}_2=\vec{v}_1 \Rightarrow \begin{bmatrix}-4&-4\\4&4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-4\\4\end{bmatrix}\triangleq\vec{v}_1$$

Note $\vec{v}_1$ is an eigenvector for $\lambda=5$.

We have

$$\vec{x}_1(t)=\vec{v}_1 e^{\lambda t}=\begin{bmatrix}-4\\4\end{bmatrix}e^{5t}$$

$$\vec{x}_2(t)=(\vec{v}_1 t+\vec{v}_2)e^{5t}=\left(\begin{bmatrix}-4t\\4t\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\right)e^{5t}$$

Remarks: Then the general solution

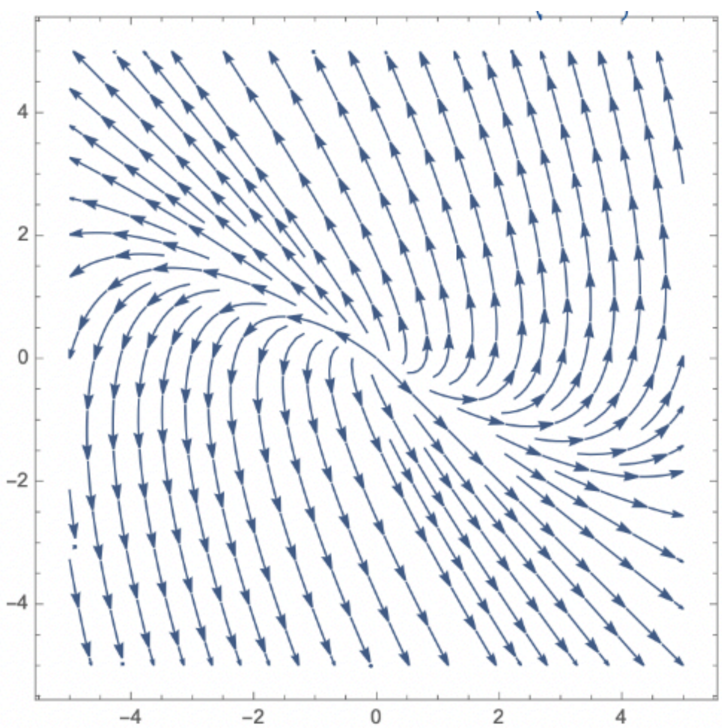
$$\vec{x}(t)=c_1\begin{bmatrix}-4\\4\end{bmatrix}e^{5t}+c_2\begin{bmatrix}-4t+1\\4t\end{bmatrix}e^{5t}$$

Remark ④: Note $\vec{v}_1$ and $\vec{v}_2$ are not unique but related by $\boxed{(A-\lambda I)\vec{v}_2=\vec{v}_1}$. For example, given $\vec{v}_1=\begin{bmatrix}-1\\1\end{bmatrix}$ we should find $\vec{v}_2$ s.t.

$$(A-5I)\vec{v}_2=\vec{v}_1 \Rightarrow \begin{bmatrix}-4&-4\\4&4\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

In Matlab, we use

```
[x,y] = meshgrid(-3:0.3:3,-3:0.3:3);
f1 = x -4*y;
f2 = 4*x + 9*y;
quiver(x,y,f1,f2)
```



Let $b=0$, then $a=1/4$.

$\vec{v}_2$ can be $\begin{bmatrix}\frac{1}{4}\\0\end{bmatrix}$

Example 3 . (λ with multiplicity 2 , and λ is defective)

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \mathbf{x} \tag{9}$$

ANS: Find the eigenvalues of A:

$$0 = |A-\lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{vmatrix} = (\lambda+2)(\lambda+4)+1 = \lambda^2+6\lambda+9 = (\lambda+3)^2 = 0$$

$\Rightarrow \lambda = -3$ is an eigenvalue of A with multiplicity 2.

· Check if $\lambda = -3$ is defective.

$$(A-\lambda I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} -2+3 & 1 \\ -1 & -4+3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a+b=0$. Any eigenvector corresponds to $\lambda=-3$ is a multiple of $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So $\lambda=-3$ is defective.

We apply the algorithm to find $\vec{v}_2$ and $\vec{v}_1$

We solve

$$(A-\lambda I)^2 \vec{v}_2 = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can choose $a=1$, $b=0$. and let $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We compute

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \triangleq \vec{v}_1$$

So we have

$$\vec{x}_1(t) = \vec{v}_1 e^{-3t}$$

$$\vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^{-3t}$$

The general solution is

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} t+1 \\ -t \end{bmatrix} e^{-3t}$$

Exercise 4 Find the general solution of the system in the following problem.

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x} \tag{10}$$

The solution can be found on page 341, Example 4.