Overview of the Eigenvalue Method (§5.2 and §5.5)

Constant Coeff. Homogeneous System:

Constant Coeff. Homogeneous: $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$

Solution: $\vec{\mathbf{x}} = c_1\vec{\mathbf{x}}_1 + c_2\vec{\mathbf{x}}_2 + \cdots$,
where $\vec{\mathbf{x}}_i$ are fundamental solutions
from eigenvalues & eigenvectors.
The method is described as below.

The Eigenvalue Method for Homogeneous Systems:

The number λ is called an *eigenvalue* of the matrix $\mathbf{A}$ if $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

An *eigenvector* associated with the eigenvalue λ is a nonzero vector $\mathbf{v}$ such that $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

We consider $\mathbf{A}$ to be 2×2, then the general solution is $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{x}}_1(t) + c_2\vec{\mathbf{x}}_2(t)$, with the fundamental solutions $\vec{\mathbf{x}}_1(t), \vec{\mathbf{x}}_2(t)$ found has follows.

- Distinct Real Eigenvalues. $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda_1 t}, \vec{\mathbf{x}}_2(t) = \vec{\mathbf{v}}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to remember the method*)

  If $\vec{v} = \vec{a} + i\vec{b}$ is an eigenvector associated with $\lambda = p + qi$, then

  $\vec{\mathbf{x}}_1(t) = e^{pt}\left(\vec{a}\cos qt - \vec{b}\sin qt\right)$, $\vec{\mathbf{x}}_2(t) = e^{pt}(\vec{b}\cos qt + \vec{a}\sin qt)$
- Defective Eigenvalue with multiplicity 2.
  Find nonzero $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ such that $(\mathbf{A} - \lambda\mathbf{I})^2\vec{\mathbf{v}}_2 = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$.
  Then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda t}$, $\vec{\mathbf{x}}_2(t) = (\vec{\mathbf{v}}_1 t + \vec{\mathbf{v}}_2)\, e^{\lambda t}$.

5.2 The Eigenvalue Method for Homogeneous Systems

In this section, we will talk about the method of solving the the first-order linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{1}$$

Review: Eigenvalues and Eigenvectors

Definition. Eigenvalues and Eigenvectors

The number λ is called an ***eigenvalue*** of the $n \times n$ matrix $\mathbf{A}$ provided that

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad \text{characteristic eqn for } A \tag{2}$$

An ***eigenvector*** associated with the eigenvalue λ is a nonzero vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, so that

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}. \tag{3}$$

Example Find the eigenvalues and eigenvectors of the given matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \tag{4}$$

ANS: The characteristic eqn for A is

$$0 = |A - \lambda I| = \left| \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$$

Thus we have two distinct eigenvalues $\lambda_1 = 5$, $\lambda_2 = -2$.

Case 1. $\lambda_1 = 5$. Substitute $\lambda_1 = 5$ into $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$.

$$\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$$

We have

$$\left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -a + 2b = 0 & ① \\ 3a - 6b = 0 & ② \end{cases}$$

Note $-3 \times ① = ②$

We can choose $b = 1$, then $a = 2$.

Rmk①: You can also choose $a = 1$, then $b = \frac{1}{2}$.

Then $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponds to $\lambda_1 = 5$.

Rmk ②: Note for any constant $c \neq 0$, $c\vec{v}_1 = \begin{bmatrix} 2c \\ c \end{bmatrix}$ is also an eigenvector corresponds to $\lambda_1 = 5$.

Case 2: $\lambda_2 = -2$, We solve $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$\begin{bmatrix} a \\ b \end{bmatrix}$

We have

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 6a + 2b = 0 & ③ \\ 3a + b = 0 & ④ \end{cases}$$

Note $③ = 2 \times ④$

Let's choose $a = 1$, then $b = -3$.

Thus $\vec{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector corresponds to $\lambda_2 = -2$.

Theorem 1 Eigenvalue Solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Let λ be an eigenvalue of the (constant) coefficient matrix $\mathbf{A}$ of the first-order linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{5}$$

If $\mathbf{v}$ is an eigenvector associated with λ, then

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} \tag{6}$$

is a nontrivial solution of the system.

Idea of the proof: Assume $\vec{x}(t) = \vec{v}e^{\lambda t}$ is a solution for (5) for some $\lambda, \vec{v}$.

Then $\vec{x}'(t) = \underline{\vec{v}\lambda \cancel{e^{\lambda t}}} = A\vec{x} = \underline{A\vec{v}\cancel{e^{\lambda t}}}$

$\Rightarrow A\vec{v} = \lambda\vec{v}$

Thus λ is an eigenvalue for A and $\vec{v}$ is the correspond eigenvector.

The Eigenvalue Method

To solve the $n \times n$ homogeneous constant-coefficient system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we have the following steps:

1. Solve the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ for the matrix $\mathbf{A}$ for the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix $\mathbf{A}$.

2. Find n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ associated with these eigenvalues by solving $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$.

3. Note step 2 is not always possible. If it is, then we get n linearly independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \cdots, \mathbf{x}_n(t) = \mathbf{v}_n e^{\lambda_n t} \qquad (1) \tag{7}$$

In this case the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{8}$$

of these n solutions.

We have the following cases for the eigenvalues:

1. Distinct Real Eigenvalues
2. Complex Eigenvalues
3. Repeated Eigenvalues (will be covered in §5.5)

Case 1. Distinct Real Eigenvalues

If the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real and distinct, then we substitute each of them in turn in $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ and solve for the associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. In this case it can be proved that the particular solution vectors given in (1) are always linearly independent.

Example 1 Apply the eigenvalue method to find a general solution of the given system. Then use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

$$\begin{aligned} x_1' &= 2x_1 + 3x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned} \Leftrightarrow \underbrace{\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}}_{\vec{x}'(t)} = \underbrace{\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\vec{x}(t)} \qquad (9)$$

ANS: Step 1. Find the eigenvalues for A.

$$0 = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4$$

$$\Rightarrow (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda_1 = -1 \text{ and } \lambda_2 = 4$$

Step 2. Find eigenvectors

- Case 1. $\lambda_1 = -1$. We solve $(A - \lambda_1 I)\underbrace{\vec{v}_1}_{\begin{bmatrix} a \\ b \end{bmatrix}} = \vec{0}$

$$\Rightarrow \begin{bmatrix} 2+1 & 3 \\ 2 & 1+1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 3a + 3b = 0 \\ 2a + 2b = 0 \end{cases} \Rightarrow a + b = 0$$

Let $a = 1$, then $b = -1$.

So $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corr. to $\lambda_1 = -1$.

- Case 2. $\lambda_2 = 4$. We solve $(A - \lambda_2 I)\underbrace{\vec{v}_2}_{\begin{bmatrix} a \\ b \end{bmatrix}} = \vec{0}$

$$\Rightarrow \begin{bmatrix} 2-4 & 3 \\ 2 & 1-4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -2a + 3b = 0 \\ 2a - 3b = 0 \end{cases} \Rightarrow 2a - 3b = 0$$

Let $a = 3$, then $b = 2$.

So $\vec{r}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector corresponds to $\lambda_2 = 4$.

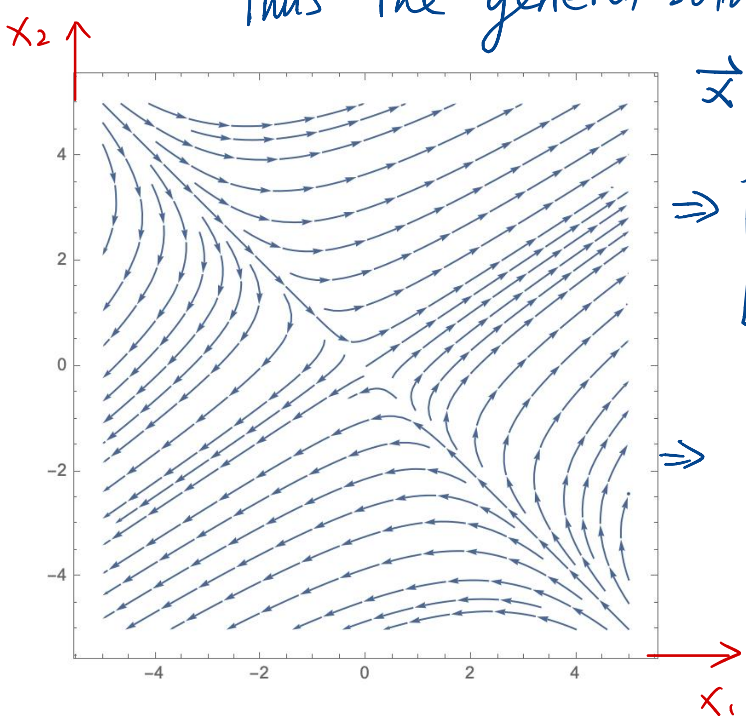
Step 3. We have

$$\vec{x}_1(t) = \vec{r}_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \quad \text{and}$$

$$\vec{x}_2(t) = \vec{v}_2 e^{\lambda_2 t} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} \quad \text{are two}$$

linearly independent solutions to the given eqn.

Thus the general solution



$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}$$

$$\Rightarrow \begin{cases} x_1(t) = c_1 e^{-t} + 3c_2 e^{4t} \\ x_2(t) = -c_1 e^{-t} + 2c_2 e^{4t} \end{cases}$$

$\frac{dx_2}{dx_1}$

In Matlab, we use

```
[x,y] = meshgrid(-3:0.3:3,-3:0.3:3);
f1 = 2*x + 3*y;
f2 = 2*x + y;
quiver(x,y,f1,f2)
```

Case 2. Complex Eigenvalues

Summary:

Assume we have complex eigenvalues $\lambda = p + qi$, $\bar{\lambda} = p - qi$.

If $\mathbf{v}$ is an eigenvector associated with $\lambda = p + qi$, then $\mathbf{v}$ can be writien as $\mathbf{v} = \mathbf{a} + i\mathbf{b}$.

Then we have the solution

Recall

$e^{(A+Bi)t} = e^{At}(\cos Bt + i\sin Bt)$

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} = (\mathbf{a} + i\mathbf{b})e^{(p+qi)t} \tag{10}$$

$$\Rightarrow \quad \mathbf{x}(t) = \underbrace{e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt)}_{\vec{x}_1(t)} + i\underbrace{e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt)}_{\vec{x}_2(t)} \tag{11}$$

Then we get the real valued solutions

$$\begin{cases} \mathbf{x}_1(t) = \mathrm{Re}(\mathbf{x}(t)) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt) \\ \mathbf{x}_2(t) = \mathrm{Im}\left(\mathbf{x}(t)\right) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt) \end{cases} \tag{12}$$

Example 2 Apply the eigenvalue method to find a general solution of the given system. Find also the corresponding particular solution to the given initial value problem. Then use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

$$\begin{aligned} x_1' &= 4x_1 - 3x_2 \\ x_2' &= 3x_1 + 4x_2 \end{aligned} \qquad A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \tag{13}$$

ANS: • Find the eigenvalue of A

$(-a)^2 = a^2$

$$0 = |A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = (\lambda-4)^2 + 9 = 0$$

$= (-(\lambda-4))^2 + 9$

$$\Rightarrow (\lambda-4)^2 = -9 \Rightarrow \lambda - 4 = \pm 3i \Rightarrow \lambda = 4 \pm 3i$$

• Consider $\lambda = 4+3i$. We want to find the corresponding eigenvector $\vec{v}$.

Let $(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} 4-(4+3i) & -3 \\ 3 & 4-(4+3i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} -3i\,a - 3b = 0 \\ 3a - 3ib = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -ia - b = 0 & ① \\ a - ib = 0 & ② \end{cases} \qquad \text{Note } i \times ① = ②$$

We can assume $a = 1$. then ② $\Rightarrow 1 - ib = 0$

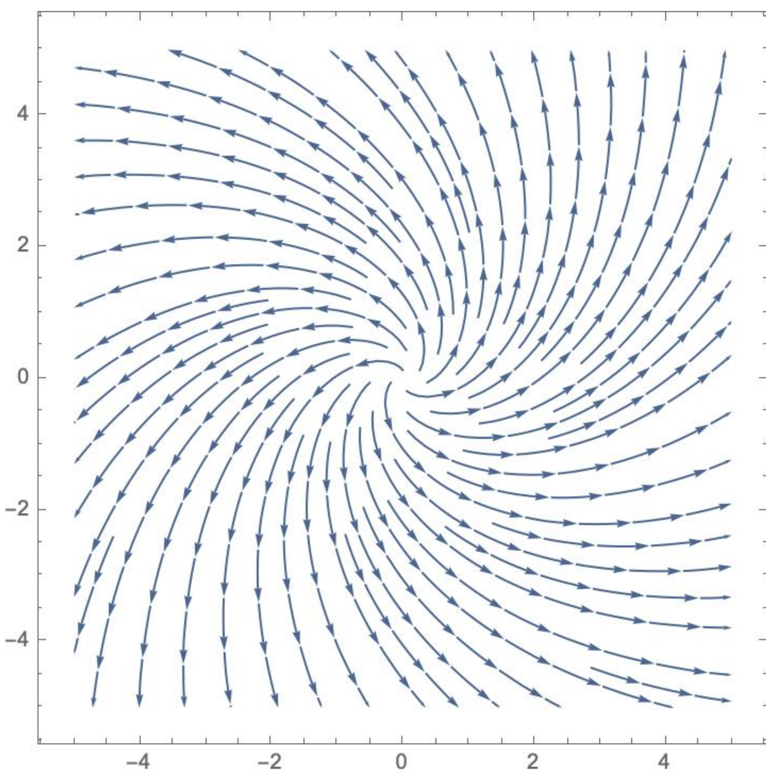
$\Rightarrow i\,ib = i \cdot 1 \Rightarrow -b = i \Rightarrow b = -i$

Then $\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector corr. to $\lambda = 4+3i$.

Then $\vec{x}(t) = \vec{v}e^{\lambda t}$ is a solution to the eqn.

Then $\vec{x}(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(4+3i)t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{4t}(\cos 3t + i\sin 3t)$

$$= \begin{bmatrix} e^{4t}\cos 3t + i\,e^{4t}\sin 3t \\ -ie^{4t}\cos 3t - \overset{(-1)}{\cancel{i^2}}\, e^{4t}\sin 3t \end{bmatrix} = \underbrace{e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}}_{\vec{x}_1(t)} + i\,\underbrace{e^{4t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}}_{\vec{x}_2(t)}$$



Note $\vec{x}_1(t)$ & $\vec{x}_2(t)$ are two linearly independent real-valued solutions. Thus the general solution is

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}$$

Example 3 In the following questions, the eigenvalues of the coefficient matrix can be found by inspection and factoring. Apply the eigenvalue method to find a general solution of each system.

(1) Similar question to the **Sec 5.2 Handwritten HW. #24** in the book

A short video has been uploaded discussing this case in the course content lecture videos.

$$\begin{aligned} x_1' &= 5x_1 + 5x_2 + 2x_3; \\ x_2' &= -6x_1 - 6x_2 - 5x_3; \\ x_3' &= 6x_1 + 6x_2 + 5x_3 \end{aligned} \qquad \vec{x}\,' = A\vec{x} \qquad \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{14}$$

ANS: · Find the eigenvalues of A:

$$0 = |A - \lambda I| = \begin{vmatrix} 5-\lambda & 5 & 2 \\ -6 & -6-\lambda & -5 \\ 6 & 6 & 5-\lambda \end{vmatrix} = (5-\lambda)\cdot\begin{vmatrix} -6-\lambda & -5 \\ 6 & 5-\lambda \end{vmatrix} - (-6)\cdot\begin{vmatrix} 5 & 2 \\ 6 & 5-\lambda \end{vmatrix}$$

$$+6\cdot\begin{vmatrix} 5 & 2 \\ -6-\lambda & -5 \end{vmatrix} = (5-\lambda)\left((-6-\lambda)(5-\lambda)+30\right) + 6\cdot\left(5(5-\lambda)-12\right)$$

$$+6\cdot\left(-25-2\cdot(-6-\lambda)\right) = -\lambda^3 + 4\lambda^2 - 13\lambda = -\lambda(\lambda^2 - 4\lambda + 13) = 0$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_{2,3} = \frac{4 \pm \sqrt{16 - 4\times 13}}{2} = 2 \pm 3i$$

· $\lambda_1 = 0$. we solve $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$, $\vec{v}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$(A - \lambda_1 I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row Reduction

$$\left[\begin{array}{ccc|c} 5 & 5 & 2 & 0 \\ -6 & -6 & -5 & 0 \\ 6 & 6 & 5 & 0 \end{array}\right] \xrightarrow{+} \left[\begin{array}{ccc|c} 5 & 5 & 2 & 0 \\ -6 & -6 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{+(\times 5)} \left[\begin{array}{ccc|c} 5 & 5 & 2 & 0 \\ -1 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} c = 0 \\ a + b = 0 \end{cases}$$

If we assume, $a=1$, then $b=-1$.

Then $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector corresponds to the eigenvalue $\lambda_1 = 0$.

- And $\vec{x}_1(t) = \vec{v}_1 e^{\lambda_1 t} = \vec{v}_1 e^{0t} = \vec{v}_1$

- $\lambda_2 = 2+3i$. we solve

$$(A-\lambda_2 I)\vec{v}_2 = \vec{0} \Rightarrow \begin{bmatrix} 5-(2+3i) & 5 & 2 \\ -6 & -6-(2+3i) & -5 \\ 6 & 6 & 5-(2+3i) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row Reduction:

$$\left[\begin{array}{ccc|c} 3-3i & 5 & 2 & 0 \\ -6 & -8-3i & -5 & 0 \\ 6 & 6 & 3-3i & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 3-3i & 5 & 2 & 0 \\ -6 & -8-3i & -5 & 0 \\ 0 & -2-3i & -2-3i & 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 3-3i & 5 & 2 & 0 \\ -6 & -8-3i & -5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 3-3i & 3 & 0 & 0 \\ -6 & -3-3i & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1-i & 1 & 0 & 0 \\ -2 & -(1+i) & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[\text{add to 2nd row}]{\text{1st row} \times (1+i)} \begin{bmatrix} 1-i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} (1+i)a + b = 0 \\ b + c = 0 \end{cases}$$

Let $c = 2$, then $b = -2$.

and $a = \dfrac{-b}{1-i} = \dfrac{2}{1-i} \cdot \dfrac{1+i}{1+i} = 1+i$

Then $\vec{V}_2 = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}$ is an eigenvector corresponds to

$\lambda_2 = 2+3i$

Then $\vec{V}_2 e^{\lambda_2 t}$ is a solution to $\vec{x}' = Ax$

$$\vec{V}_2 e^{\lambda_2 t} = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix} e^{(2+3i)t} = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix} e^{2t}(\cos 3t + i\sin 3t)$$

$$= \begin{bmatrix} e^{2t}(\cos 3t + i\sin 3t) + ie^{2t}\cos 3t - e^{2t}\sin 3t \\ -2e^{2t}\cos 3t - 2ie^{2t}\sin 3t \\ 2e^{2t}\cos 3t + 2ie^{2t}\sin 3t \end{bmatrix}$$

$$= \underbrace{e^{2t}\begin{bmatrix} \cos 3t - \sin 3t \\ -2\cos 3t \\ 2\cos 3t \end{bmatrix}}_{\vec{x}_2(t)} + i\underbrace{e^{2t}\begin{bmatrix} \sin 3t + \cos 3t \\ -2\sin 3t \\ 2\sin 3t \end{bmatrix}}_{\vec{x}_3(t)}$$

Therefore,

$$\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) + C_3\vec{x}_3(t)$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \vec{v}_1 e^{\overset{\lambda_1}{0}\cdot t} + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

$$= c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \cos 3t - 3\sin 3t \\ -2\cos 3t \\ 2\cos 3t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ -2\sin 3t \\ 2\sin 3t \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1(t) = c_1 + c_2 e^{2t}(\cos 3t - 3\sin 3t) + c_3 e^{2t}(\cos 3t + \sin 3t) \\ x_2(t) = -c_1 + c_2 e^{2t}\cdot(-2\cos 3t) + c_3 e^{2t}\cdot(-2\sin 3t) \\ x_3(t) = 2c_2 e^{2t}\cos 3t + 2c_3 e^{2t}\sin 3t \end{cases}$$

(2) **Exercise.** $x_1' = 4x_1 + x_2 + x_3$
$x_2' = x_1 + 4x_2 + x_3$
$x_3' = x_1 + x_2 + 4x_3$

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 4-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 4-\lambda & 1 \end{vmatrix}$$

$$= (4-\lambda)\left((4-\lambda)^2 - 1\right) - (4-\lambda-1) + (1-(4-\lambda))$$

$$= (4-\lambda)(\lambda^2 - 8\lambda + 15) - (-\lambda+3) + (\lambda-3)$$

$$0 = \lambda^3 - 12\lambda^2 + 45\lambda - 54$$

Notice that $\lambda = 3$ is a solution. So we try to factor

$$\begin{array}{r|l} & \lambda^2 - 9\lambda + 18 \\ \hline \lambda - 3 & \lambda^3 - 12\lambda^2 + 45\lambda - 54 \\ & \lambda^3 - 3\lambda^2 \\ \hline & -9\lambda^2 + 45\lambda \\ & -9\lambda^2 + 27\lambda \\ \hline & 18\lambda - 54 \end{array}$$

Thus $\lambda^3 - 12\lambda^2 + 45\lambda - 54 = (\lambda-3)(\lambda^2 - 9\lambda + 18)$

$= (\lambda-3)(\lambda-3)(\lambda-6) = 0$

$\Rightarrow \lambda = 3, 3, 6.$

- When $\lambda = 3$, we solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a + b + c = 0$$

If $c = 0$, we have $a + b = 0$, We can assume $a = 1$, $b = -1$.

So $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector corresponds to the eigenvalue $\lambda = 3$.

If $b=0$, then $a+c=0$. We can assume $a=1$, $c=-1$.

So $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is another eigenvector associated to $\lambda=3$.

This means $\lambda_1=3$ is complete by the §5.5.

If $\lambda=6$, we solve.

$$\begin{bmatrix} 4-6 & 1 & 1 \\ 1 & 4-6 & 1 \\ 1 & 1 & 4-6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & -3 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Thus $\begin{cases} -b+c=0 \\ a-c=0 \end{cases}$ If $c=1$, then $a=b=1$

Thus $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigonvectors corresponds to $\lambda=6$

Thus the general solution is

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

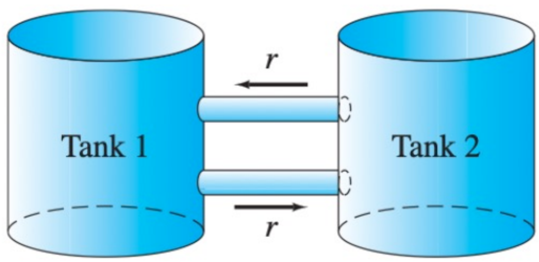
$$= c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{6t} .$$

Example 4 (Similar question to **Handwritten HW Section 5.2. #29**)

The amounts $x_1(t)$ and $x_2(t)$ of salt in the two brine tanks of the following figure satisfy the differential equations

$$\frac{dx_1}{dt} = -k_1x_1 + k_2x_2, \qquad \frac{dx_2}{dt} = k_1x_1 - k_2x_2, \tag{16}$$

where $k_i = \dfrac{r}{V_i}$ as usual. Assume $V_1 = 25$ (gal) and $V_2 = 40$ (gal). Solve for $x_1(t)$ and $x_2(t)$, assuming that $r = 10$ (gal/min), $x_1(0) = 15$ (lb), and $x_2(0) = 0$. Then construct a figure showing the graphs of $x_1(t)$ and $x_2(t)$.



ANS: We have $k_1 = \frac{r}{V_1} = \frac{10}{25} = \frac{2}{5}$, $k_2 = \frac{r}{V_2} = \frac{10}{40} = \frac{1}{4}$

Then $\begin{cases} x_1' = -\frac{2}{5}x_1 + \frac{1}{4}x_2 \\ x_2' = \frac{2}{5}x_1 - \frac{1}{4}x_2 \end{cases}$ Then $A = \begin{bmatrix} -\frac{2}{5} & \frac{1}{4} \\ \frac{2}{5} & -\frac{1}{4} \end{bmatrix}$

· First we find the eigenvalues for A.

$$0 = |A - \lambda I| = \begin{vmatrix} -\frac{2}{5}-\lambda & \frac{1}{4} \\ \frac{2}{5} & -\frac{1}{4}-\lambda \end{vmatrix} = \left[\left(\frac{2}{5}+\lambda\right)\left(\frac{1}{4}+\lambda\right) - \frac{1}{10}\right]^{\times 20} = 0^{\times 20}$$

$\Rightarrow (2+5\lambda)(1+4\lambda) - 2 = 0 \Rightarrow 20\lambda^2 + 13\lambda = 0$

$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = -\frac{13}{20}$

· $\lambda_1 = 0$. We solve $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\Rightarrow \begin{bmatrix} -\frac{2}{5} & \frac{1}{4} \\ \frac{2}{5} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $b = 8$, then

$-\frac{2}{5}a + \frac{1}{4}\overset{8}{b} = 0 \Rightarrow a = 5.$

Then $\vec{v}_1 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ is an eigenvector corresponds to $\lambda_1 = 0$.

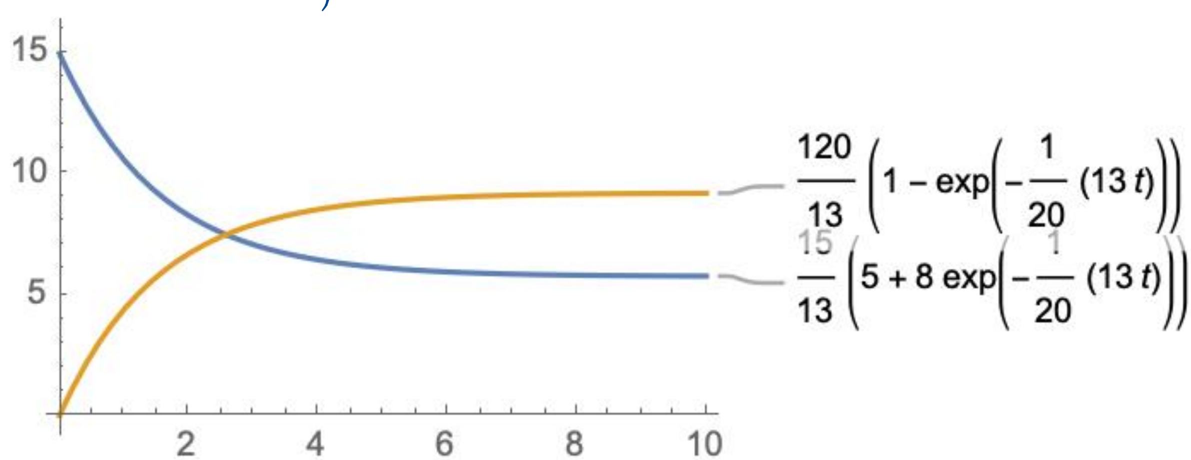
· $\lambda_2 = -\frac{13}{20}$ we solve.

$$(A - \lambda_2 I)\vec{v}_2 = \vec{0} \Rightarrow \begin{bmatrix} -\frac{2}{5} + \frac{13}{20} & \frac{1}{4} \\ \frac{2}{5} & -\frac{1}{4} + \frac{13}{20} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a + b = 0$ Let $a = 1$, then $b = -1$.

Thus $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponds to $\lambda_2 = -\frac{13}{20}$.



Thus $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 5 \\ 8 \end{bmatrix} \cancel{e}^{1} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-\frac{13}{20}t}$

As $x_1(0) = 15$, $x_2(0) = 0$, we have.

$$\begin{bmatrix} 15 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ 8 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} c_1 = \frac{15}{13} \\ c_2 = \frac{120}{13} = \frac{15 \times 8}{13} \end{cases}$$

Thus

$$\begin{cases} x_1(t) = \frac{15}{13}\left(5 + 8e^{-\frac{13}{20}t}\right) \\ x_2(t) = \frac{120}{13}\left(1 - e^{-\frac{13}{20}t}\right) \end{cases}$$