5.1 Matrices and Linear Systems

Review of Matrix Notation and Terminology

An $m \times n$ matrix $\mathbf{A}$ is a rectangular array of mn numbers (or elements) arranged in m (horizontal) rows and n (vertical) columns:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{pmatrix}$$

Two $m \times n$ matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are said to be equal if corresponding elements are equal. We have

$$\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

$$c\mathbf{A} = \mathbf{A}c = [ca_{ij}]$$

We have

- $A + B = B + A$
- $(A+B) + C = A + (B+C)$
- $c(A+B) = cA + cB$
- $(c+d)A = cA + dA$

The **transpose $\mathbf{A}^T$** of the $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the $n \times m$ matrix whose j th column is the j th row of $\mathbf{A}$

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Matrix Multiplication

If

$$\mathbf{a} = \begin{bmatrix} a_1 \, a_2 \, \cdots \, a_p \end{bmatrix} \quad \text{and } \mathbf{b} = \begin{bmatrix} b_1 \, b_2 \, \cdots \, b_p \end{bmatrix}^T$$

then the **scalar product** $\mathbf{a} \cdot \mathbf{b}$ is defined as follows:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^{p} a_k b_k = a_1 b_1 + a_2 b_2 + \cdots + a_p b_p$$

The product $\mathbf{AB}$ of two matrices is defined only if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$. If $\mathbf{A}$ is an $m \times p$ matrix and B is a $p \times n$ matrix, then their product $\mathbf{AB}$ is the $m \times n$ matrix

$$\mathbf{C} = \begin{bmatrix} c_{ij} \end{bmatrix}$$

where c_{ij} is the scalar product of the ith row vector $\mathbf{a}_i$ of $\mathbf{A}$ and the jth column vector $\mathbf{b}_j$ of $\mathbf{B}$. Thus

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_i \cdot \mathbf{b}_j \end{bmatrix}$$

If $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{ij} \end{bmatrix}$, then we have

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

For the computation by hand, it is easy to remember by visualizing the picture

$$\begin{array}{r} \\ \\ \\ \mathbf{a}_i \longrightarrow \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{array}{c} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} \\ \uparrow \\ \mathbf{b}_j \end{array},$$

which shows that one forms the dot product of the row vector $\mathbf{a}_i$ with the column vector $\mathbf{b}_j$ to obtain the element c_{ij} in the ith row and the j th column of $\mathbf{AB}$.

Inverse Matrices

$a \in \mathbb{R}, \quad a \cdot 1 = 1 \cdot a = a$

$a \cdot b = b \cdot a = 1$

then we call b an inverse of a, which is $\frac{1}{a} = a^{-1}$

The **identity** matrix of order n is the square matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

We have

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

If $\mathbf{A}$ is a square matrix, then an inverse of $\mathbf{A}$ is a square matrix $\mathbf{B}$ of the same order as $\mathbf{A}$ such that both

$$\mathbf{AB} = \mathbf{I} \text{ and } \mathbf{BA} = \mathbf{I}$$

We denote such $\mathbf{B}$ by $\mathbf{A}^{-1}$.

Rmk: Note B may not exist! E.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

When A^{-1} exists?

In linear algebra it is proved that $\mathbf{A}^{-1}$exists if and only if the determinant $\det(\mathbf{A})$ of the square matrix $\mathbf{A}$ is nonzero. If so, the matrix $\mathbf{A}$ is said to be **nonsingular**; if $\det(\mathbf{A}) = 0$, then $\mathbf{A}$ is called a **singular** matrix.

Example 1 Find $\mathbf{AB}$ and $\mathbf{BA}$ given

$$\mathbf{A} = \begin{pmatrix} 5 & 3 & 4 \\ 3 & -2 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}$$

ANS: $AB = \begin{pmatrix} 5 & 3 & 4 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 5\times1 + 3\times4 + 2\times4 & 5\times2 + 3\times5 + 4\times3 \\ 3\times1 - 2\times4 + 1\times2 & 3\times2 - 2\times5 + 1\times3 \end{pmatrix}$

$= \begin{pmatrix} 25 & 37 \\ -3 & -1 \end{pmatrix}$

$BA = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 3 & 4 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 5+6 & 3-4 & 4+2 \\ 20+15 & 12-10 & 16+5 \\ 10+9 & 0 & 8+3 \end{pmatrix}$

$= \begin{pmatrix} 11 & -1 & 6 \\ 35 & 2 & 21 \\ 19 & 0 & 11 \end{pmatrix}$ Note $AB \neq BA$

Matrix-Valued Functions

A **matrix-valued function** is a matrix such as

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots a_{2n}(t) \\ \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots a_{nn}(t) \end{pmatrix}$$

in which each entry is a function of t.

We say that the matrix function $\mathbf{A}(t)$ is **continuous** (or **differentiable**) at a point (or on an interval) if each of its elements has the same property. The **derivative** of a differentiable matrix function is defined by elementwise differentiation:

$$\mathbf{A}'(t) = \frac{d\mathbf{A}}{dt} = [\frac{da_{ij}}{dt}]$$

Example 2 Let A and B be the matrices as in Example 1. Let

$$\mathbf{x} = \begin{pmatrix} e^{-2t} \\ 3t \end{pmatrix}_{2\times1} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} t^3 \\ \tan t \\ \sin t \end{pmatrix}_{3\times1}$$

Find $\mathbf{Ay}$ and $\mathbf{Bx}$. Are the products $\mathbf{Ax}$ and $\mathbf{By}$ well-defined?

ANS: $A\vec{y} = \begin{pmatrix} 5 & 3 & 4 \\ 3 & -2 & 1 \end{pmatrix}_{2\times3} \begin{pmatrix} t^3 \\ \tan t \\ \sin t \end{pmatrix}_{3\times1} = \begin{pmatrix} 5t^3 + 3\tan t + 4\sin t \\ 3t^3 - 2\tan t + \sin t \end{pmatrix}_{2\times1}$

$B\vec{x} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}_{3\times2} \begin{pmatrix} e^{-2t} \\ 3t \end{pmatrix}_{2\times1} = \begin{pmatrix} e^{-2t} + 6t \\ 4e^{-2t} + 15t \\ 2e^{-2t} + 9t \end{pmatrix}_{3\times1}$

The product $A\vec{x}$ and $B\vec{y}$ are not well-defined. since A is a 2×3 matrix but x is a 2×1 matrix, and,

does not equal

B is a 3×2 matrix, but y is a 3×1 matrix

Example 3 Find $\mathbf{A}'$ if

$$\mathbf{A}(t) = \begin{pmatrix} 3t & t^2 \\ t^3 & 3+t^4 \end{pmatrix}$$

ANS: $A' = \begin{pmatrix} (3t)' & (t^2)' \\ (t^3)' & (3+t^4)' \end{pmatrix} = \begin{pmatrix} 3 & 2t \\ 3t^2 & 4t^3 \end{pmatrix}$

First-Order Linear Systems

We discuss here the general system of n first-order linear equations

$$\begin{aligned} x_1' &= p_{11}x_1 + p_{12}x_2 + \cdots p_{1n}x_n + f_1(t), \\ x_2' &= p_{21}x_1 + p_{22}x_2 + \cdots p_{2n}x_n + f_2(t), \\ x_3' &= p_{31}x_1 + p_{32}x_3 + \cdots p_{3n}x_n + f_3(t), \\ &\vdots \\ x_n' &= p_{n1}x_1 + p_{n2}x_2 + \cdots p_{nn}x_n + f_n(t), \end{aligned} \Leftrightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ p_{31} & p_{32} & \cdots & p_{3n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

If we introduce the coefficient matrix

$$\mathbf{P}(t) = \left[p_{ij}(t)\right]$$

and the column vectors

$$\mathbf{x} = \left[x_i\right] \quad \text{and} \quad \mathbf{f}(t) = \left[f_i(t)\right]$$

Then the above system takes the form of a single matrix equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t) \tag{1}$$

A solution of Eq. (1) on the open interval I is a column vector function $\mathbf{x}(t) = \left[x_i(t)\right]$ such that the component functions of $\mathbf{x}$ satisfy the above system identically on I.

Example 4 Write the given system in the for $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$.

(1) $$\begin{aligned} x' &= x + 3y + 2e^t, \\ y' &= 4x - y - t^2 \end{aligned}$$

ANS: We have $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $P(t) = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$. $\vec{f}(t) = \begin{bmatrix} 2e^t \\ -t^2 \end{bmatrix}$

Then $$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}'}_{\vec{x}'(t)} = \underbrace{\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}}_{P(t)} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 2e^t \\ -t^2 \end{bmatrix}}_{\vec{f}(t)}$$

(2) $x' = 2x - 3y,$ $= 2x - 3y + 0 \cdot z$

$y' = x + y + 2z,$

$z' = 5y - 7z$ $= 0 \cdot x + 5y - 7z$

ANS: We have $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $P = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 0 & 5 & -7 \end{bmatrix}$, $\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 0 & 5 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

To solve the Eq. (1) in general, we consider first the the **associated homogeneous equation**

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} \tag{2}$$

We expect it to have n solutions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ that are independent in some appropriate sense, and such that every solution of Eq. (2) is a linear combination of these n particular solutions.
Given n solutions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ of Eq. (2), we write

$$\mathbf{x}_j(t) = \begin{pmatrix} x_{1j}(t) \\ \vdots \\ x_{ij}(t) \\ \vdots \\ x_{nj}(t) \end{pmatrix} \tag{3}$$

Theorem 1 Principle of Superposition

Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be n solutions of the homogeneous linear equation in (2) on the open interval I. If $c_1, c_2, \cdots c_n$ are constants, then the linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

is also a solution of Eq. (2) on I.

Independence and General Solutions

The vector-valued functions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \cdots, c_n$ not all zero such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0}$$

for all t in I. Otherwise, they are **linearly independent**.

Just as in the case of a single nth-order equation, there is a Wronskian determinant that tells us whether or not n given solutions of the homogeneous equation in (1) are linearly dependent. If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are such solutions, then their Wronskian is the $n \times n$ determinant

$$W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots x_{2n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots x_{nn}(t) \end{vmatrix}$$

using the notation in (3) for the components of the solutions.

Theorem 2 Wronskians of Solutions

Suppose that $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are n solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I. Suppose also that $\mathbf{P}(t)$ is continuous on I. Let

$$W = W(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$$

Then

- If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linearly dependent on I, then $W = 0$ at every point of I.
- If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linearly independent on I, then $W \neq 0$ at each point of I.
  Thus there are only two possibilities for solutions of homogeneous systems: Either $W = 0$ at every point of I, or $W \neq 0$ at no point of I.

Theorem 3 General Solutions of Homogeneous Systems

Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I, where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I, then there exist numbers $c_1, c_2, \cdots c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

for all t in I.

Example 5 In the following question, first verify that the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system

Step 1 Step 2 Step 3

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\mathbf{x}; \quad \mathbf{x}_1 = e^{2t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = e^{-5t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

ANS: Step 1.

For $\vec{x}_1$, $LHS = \vec{x}_1{}' = \begin{pmatrix} 6 e^{2t} \\ 4 e^{2t} \end{pmatrix} \overset{\checkmark}{=} RHS = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 6 e^{2t} \\ 4 e^{2t} \end{pmatrix}$

For $\vec{x}_2$, $LHS = \vec{x}_2{}' = \begin{pmatrix} -5e^{-5t} \\ -15 e^{-5t} \end{pmatrix} \overset{\checkmark}{=} RHS = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} = \begin{pmatrix} -5e^{-5t} \\ -15e^{-5t} \end{pmatrix}$

Step 2. We compute the Wronskian of $\vec{x}_1(t)$ $\vec{x}_2(t)$:

$$W(t) = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 9e^{-3t} - 2e^{-3t} = 7e^{-3t} \neq 0$$

By Thm 2, we know $\vec{x}_1(t)$ $\vec{x}_2(t)$ are linearly independent.

Step 3. The general solution is (by Thm 3)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = c_1 e^{2t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad \vec{x}(t) = \begin{pmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{pmatrix}$$

$$\Rightarrow \begin{cases} x(t) = 3c_1 e^{2t} + c_2 e^{-5t} \\ y(t) = 2c_1 e^{2t} + 2c_2 e^{-5t} \end{cases}$$

Example 6 Find a particular solution of the linear system that satisfies the given initial conditions.

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\mathbf{x}; \quad \mathbf{x}_1 = e^{2t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = e^{-5t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \cancel{x_1}\,x(0) = 8, \quad \cancel{x_2}\,y(0) = 0$$

ANS: By Example 5, we know

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \vec{x}(t) = \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{bmatrix}$$

Since

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3c_1 e^{2\cdot 0} + c_2 e^{-5\cdot 0} \\ 2c_1 e^{2\cdot 0} + 3c_2 e^{-5\cdot 0} \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2 \\ 2c_1 + 3c_2 \end{bmatrix} \Rightarrow \begin{cases} 3c_1 + c_2 = 8 \\ 2c_1 + 3c_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{24}{7} \\ c_2 = -\frac{16}{7} \end{cases}$$

Thus

$$\vec{x}(t) = \begin{bmatrix} \frac{72}{7} e^{2t} - \frac{16}{7} e^{-5t} \\ \frac{48}{7} e^{2t} - \frac{48}{7} e^{-5t} \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$