Online HW #23 Question 4.

Use the elimination method to find a general solution for the given linear system, where differentiation is with respect to t.

$$\begin{aligned} 2x' + y' - 3x - 6y &= e^{-t} \\ x' + y' + 7x + 2y &= e^{t} \end{aligned}$$

Similar Question:

$$\begin{cases} 2y' - x' = x + 3y + e^{t} & (1) \\ 3x' - 4y' = x - 15y + e^{-t} & (2) \end{cases}$$

Hint: First we want to solve the above eqns for x' and y'. So we compute

$(1) \times 3 + (2)$

$$\Rightarrow 6y' \cancel{- 3x'} + (\cancel{3x'} - 4y') = 3(x + 3y + e^{t}) + x - 15y + e^{-t}$$

$$\Rightarrow 2y' = 4x - 6y + 3e^{t} + e^{-t}$$

$$\Rightarrow y' = 2x - 3y + \frac{3}{2}e^{t} + \frac{e^{-t}}{2} \quad (3)$$

Substitute ③ into ①, we have

$$x' = 3x - 9y + e^{-t} + 2e^{t}$$

Thus
$$\begin{cases} x' = 3x - 9y + e^{-t} + 2e^{t} & ④ \\ y' = 2x - 3y + \frac{3}{2}e^{t} + e^{-t} & ⑤ \end{cases}$$

From ④, we know

$$y = \frac{1}{9}(-x' + 3x + e^{-t} + 2e^{t}) \quad ⑥$$

and take the derivative we have

$$y' = \frac{1}{9}(-x'' + 3x' - e^{-t} + 2e^{t}) \quad ⑦$$

Plug ⑥ ⑦ into ⑤, we have $x'' + 9x = -\frac{1}{2}(5e^{-t} + 11e^{t})$

Solving the above eqn for x (some long computation)

$$\boxed{x(t) = C_1\cos 3t + C_2\sin t - \frac{1}{4}e^{-t} - \frac{11}{20}e^{t}.}$$

Substitute $x(t)$ in 6, we have

$$\boxed{y(t) = \frac{1}{3}(C_1 - C_2)\cos 3t + \frac{1}{3}(C_1 + C_2)\sin 3t + \frac{1}{10}e^t}$$

4.2 The Method of Elimination

$$\begin{cases} x+y=0 \Rightarrow y=-x \\ 2x+3y=5 \end{cases} \Rightarrow 2x-3x=5 \Rightarrow x=-5$$

$y=5$

The method of elimination for linear differential systems is similar to the solution of a linear system of algebraic equations by a process of eliminating the unknowns one at a time until only a single equation with a single unknown remains.

Example 1 Find a general solution of the linear system.

$$x' = -3x - 4y, \textcircled{1} \quad y' = 2x + y \quad \textcircled{2}$$

ANS: From ①, we have

$$y = -\frac{1}{4}(x'+3x)$$

Then $$y' = -\frac{1}{4}(x''+3x')$$

We substitute y, y' into ②, we have

$$-\frac{1}{4}(x''+3x') = 2x - \frac{1}{4}(x'+3x)$$

$$\Rightarrow \quad x''+3x' = -8x + x' + 3x$$

$$\Rightarrow \quad x''+2x'+5x=0$$

$$r^2+2r+5=0 \Rightarrow r = \frac{-2\pm\sqrt{4-20}}{2} = -1 \pm 2i$$

Then $$x(t) = e^{-t}(C_1\cos 2t + C_2 \sin 2t)$$

Substitute this into $$y = -\frac{1}{4}(x'+3x)$$

$$\Rightarrow y(t) = \frac{1}{2}e^{-t}\left[-(C_1+C_2)\cos 2t + (C_1-C_2)\sin 2t\right]$$

Example 2 Find a general solution of the linear system.

$$x'' = -5x + 2y, \text{①} \quad y'' = 2x - 8y \text{ ②}$$

ANS: From ①, we have

$$y = \frac{1}{2}(x'' + 5x)$$

$$\Rightarrow \quad y'' = \frac{1}{2}(x^{(4)} + 5x'')$$

We substitute y, y'' into ②. we have

$$\frac{1}{2}(x^{(4)} + 5x'') = 2x - 8 \cdot \frac{1}{2}(x'' + 5x)$$

$$\Rightarrow \quad x^{(4)} + 5x'' = 4x - 8x'' - 40x$$

$$\Rightarrow \quad x^{(4)} + 13x'' + 36x = 0$$

$$\underbrace{r^4}_{= (r^2)^2} + 13r^2 + 36 = 0 \Rightarrow (r^2+4)(r^2+9) = 0$$

$$\Rightarrow r^2 = -4 \quad \text{or} \quad r^2 = -9$$

$$\Rightarrow r_{1,2} = \pm 2i \quad \& \quad r_{3,4} = \pm 3i$$

Then $x(t) = C_1 \cos 2t + C_2 \sin 2t + C_3 \cos 3t + C_4 \sin 3t$.

Subsitute x and x'' into $y = \frac{1}{2}(x'' + 5x)$, we have

$$y(t) = \frac{1}{2} C_1 \cos 2t + \frac{1}{2} C_2 \sin 2t - 2C_3 \cos 3t - 2C_4 \sin 3t.$$

Example 3 Find a general solution of the linear system.

$$x' = 2x - 3y + 2\sin 2t \quad ① \quad y' = x - 2y - \cos 2t \quad ②$$

ANS: From ①, we have

$$y = \frac{1}{3}(-x' + 2x + 2\sin 2t)$$

$$\Rightarrow \quad y' = \frac{1}{3}(-x'' + 2x' + 4\cos 2t)$$

Plug y, y' into ②, we have

$$y' = \frac{1}{3}(-x'' + 2x' + 4\cos 2t) = x - \frac{2}{3}(-x' + 2x + 2\sin 2t) - \cos 2t$$

$$\Rightarrow \quad -x'' + \cancel{2x'} + 4\cos 2t = 3x + \cancel{2x'} - 4x - 4\sin 2t - 3\cos 2t$$

$$\Rightarrow \quad -x'' + x = -4\sin 2t - 7\cos 2t$$

$$\Rightarrow \quad x'' - x = 4\sin 2t + 7\cos 2t \quad \text{(non-homogeneous)}$$

As $r^2 - 1 = 0$, $\Rightarrow r = \pm 1$

$$x_c = C_1 e^{-t} + C_2 e^{t}$$

Assume $x_p = A\cos 2t + B\sin 2t$

$$x_p'' = -4A\cos 2t - 4B\sin 2t$$

Then

$$x_p'' - x_p = -4A\cos 2t - 4B\sin 2t - A\cos 2t - B\sin 2t = 4\sin 2t + 7\cos 2t$$

$$\Rightarrow \begin{array}{l} -5A = 7 \\ -5B = 4 \end{array} \Rightarrow \begin{cases} A = -\frac{7}{5} \\ B = -\frac{4}{5} \end{cases}$$

$$x(t) = x_c + x_p = C_1 e^{-t} + C_2 e^{t} - \frac{7}{5}\cos 2t - \frac{4}{5}\sin 2t.$$

Subs this in $y = \frac{1}{3}(-x' + 2x + 2\sin 2t)$, we have

$$y(t) = C_1 e^{-t} + C_2 e^{t} - \frac{1}{5}(2\cos 2t + 4\sin 2t)$$