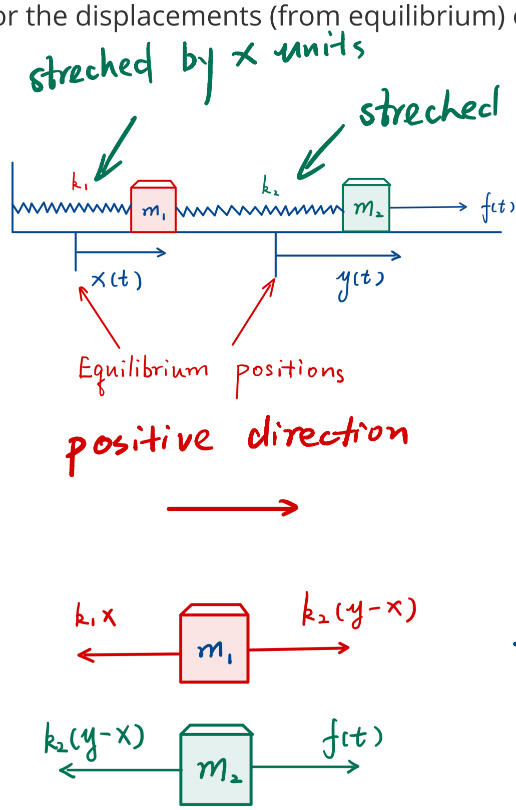
Ch4 Introduction to Systems of Differential Eqns

4.1 First-Order Systems and Applications

Example 0 Derive the equations

$$
\begin{aligned}
m_1 x'' &= -k_1 x + k_2(y-x) \\
m_2 y'' &= -k_2(y-x) + f(t)
\end{aligned}
$$

for the displacements (from equilibrium) of the two masses shown in the following figure.



By Newton's law of motion

$F = ma$

For m_1, we have

$m_1 x'' = k_2(y-x) - k_1 x$

For m_2, we have

$m_2 y'' = f(t) - k_2(y-x)$

Thus

$$
\begin{cases}
m_1 x'' = -k_1 x + k_2(y-x) \\
m_2 y'' = f(t) - k_2(y-x)
\end{cases}
$$

Consider the single nth-order equation

$$
x^{(n)} = f(t, x, x', \cdots, x^{(n-1)})
$$

We introduce the independent variables $x_1, x_2, \cdots, x_n$ as follows:

$$
x_1 = x,\ x_2 = x',\ x_3 = x'',\ \cdots, x_n = x^{(n-1)}.
$$

Then we have the following system

$$
\begin{cases}
x_1' = x_2 \\
x_2' = x_3 \\
\cdots \\
x_{n-1}' = x_n \\
x_n' = f(t, x_1, x_2, \cdots, x_n)
\end{cases}
$$

Example 1 Transform the given differential equation into an equivalent system of first-order differential equations.

$$x'' + 2x' + 26x = 34\cos 4t$$

ANS: Let $x_1 = x$, $x_2 = x_1' = x'$, So $x_2' = x'' = -26x - 2x' + 34\cos 4t$

$$= -26x_1 - 2x_2 + 34\cos 4t$$

Thus
$$\begin{cases} x_1' = x_2 \\ x_2' = -26x_1 - 2x_2 + 34\cos 4t \end{cases}$$

Example 2 Transform the given differential equation into an equivalent system of first-order differential equations.

$$x^{(4)} + 3x'' + x = e^{2t}\sin 3t$$

ANS: Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, $x_4 = x_3' = x'''$

$$x_4' = x^{(4)} = -3x'' - x - e^{2t}\sin 3t = -3x_3 - x_1 + e^{2t}\sin 3t.$$

Thus
$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = -3x_3 - x_1 + e^{2t}\sin 3t \end{cases}$$

Example 3 Transform the given differential equation into an equivalent system of first-order differential equations.

$$t^3x^{(3)} - 2t^2x'' + 3tx' + 5x = \ln t$$

ANS: Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$,

$x_3' = x^{(3)}$ So

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ t^3 x_3' = 2t^2 x_3 - 3t x_2 - 5x_1 + \ln t \end{cases}$$

$$\Updownarrow$$

$$x_3' = \frac{2}{t} x_3 - \frac{3}{t^2} x_2 - \frac{5}{t^3} x_1 + \frac{\ln t}{t^3} \quad (t \neq 0)$$

Example 4 Transform the given differential equation system into an equivalent system of first-order differential equations.

$$x'' = 3x - y + 2z, \quad y'' = x + y - 4z, \quad z'' = 5x - y - z$$

ANS: Let $x_1 = x$, $x_2 = x_1'$, then $x_2' = x''$

Let $y_1 = y$, $y_2 = y_1'$, then $y_2' = y''$

Let $z_1 = z$, $z_2 = z_1'$, then $z_2' = z''$

$$\begin{cases} x_1' = x_2 \\ x_2' = 3x_1 - y_1 + 2z_1 \\ y_1' = y_2 \\ y_2' = x_1 + y_1 - 4z_1 \\ z_1' = z_2 \\ z_2' = 5x_1 - y_1 - z_1 \end{cases}$$

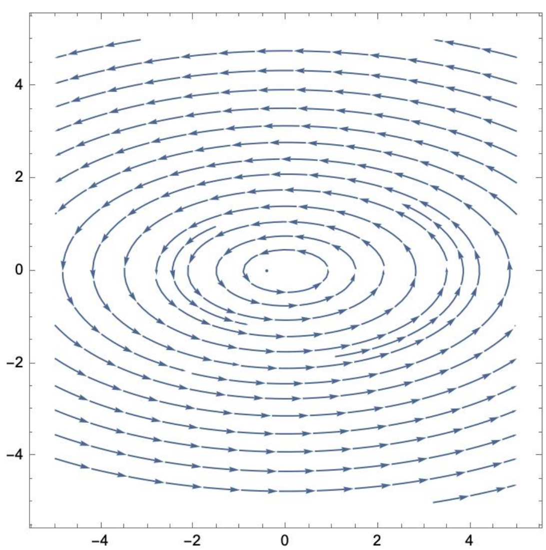
Example 5 Solve the two-dimensional system

$$x' = -2y \Rightarrow y = -\tfrac{1}{2}x' \Rightarrow y' = -\tfrac{1}{2}x''$$
$$y' = \frac{1}{2}x$$

ANS: From the first equation, we have $y = -\frac{1}{2}x'$.

Take the $\frac{d}{dt}$ both sides, we have $y' = -\frac{1}{2}x''$

Then compare with the second eqn. we have



$$y' = \boxed{-\tfrac{1}{2}x'' = \tfrac{1}{2}x} \Rightarrow -\tfrac{1}{2}x'' = \tfrac{1}{2}x \Rightarrow -x'' = x$$
$$\Rightarrow x'' + x = 0$$

$$\Rightarrow x'' + x = 0$$

$$r^2 + 1 = 0 \Rightarrow r = \pm i$$

Then $x(t) = A\cos t + B\sin t = C\cos(t-\alpha)$

Then since

$$y = -\tfrac{1}{2}x' = -\tfrac{1}{2}\left[C\cos(t-\alpha)\right]'$$

$$\Rightarrow y = +\tfrac{1}{2}C\sin(t-\alpha)$$

$$\Rightarrow \begin{cases} x(t) = C\cos(t-\alpha) \\ y(t) = \frac{1}{2}C\sin(t-\alpha) \end{cases} \Rightarrow \begin{cases} (C\cos(t-\alpha))^2 = (x(t))^2 \\ \quad + \qquad\qquad + \\ (C\sin(t-\alpha))^2 = (2y(t))^2 \end{cases}$$

In Mathematica, we can type

$$\Rightarrow \underbrace{C^2\cos^2(t-\alpha) + C^2\sin^2(t-\alpha)}_{= C^2} = x^2(t) + 4y^2(t)$$

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StreamPlot[{-2*y, (1/2)*x}, {x, -5, 5}, {y, -5, 5}]
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In Matlab, we use

$$\Rightarrow x^2 + 4y^2 = C^2$$

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[x,y] = meshgrid(-3:0.3:3,-3:0.3:3);
f1 = -2*y;
f2 = (1/2)*x;
quiver(x,y,f1,f2)
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$$\Rightarrow \frac{x^2}{C^2} + \frac{y^2}{(C/2)^2} = 1$$

Thus the solution curves (trajectories) are ellipses.

Example 6 Find the general solution of the following problem. Show that the trajectories of this system are hyperbolas.

$$x' = y, \qquad y' = 2x$$

ANS: From the first eqn, $y = x' \Rightarrow y' = x''$

Plug $y' = x''$ into the second eqn, we have

$$y' = \boxed{x'' = 2x}$$

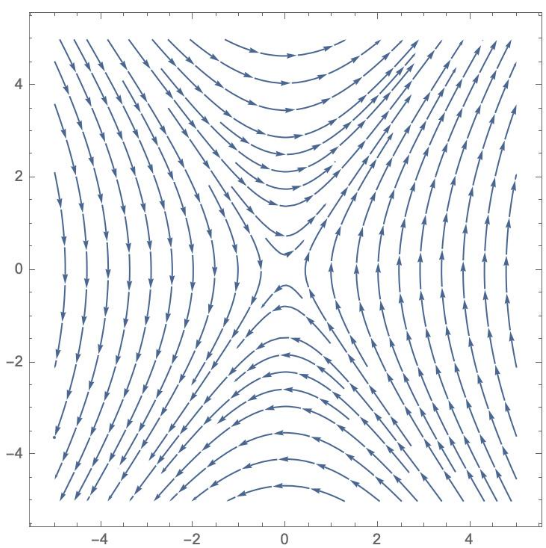
$$\Rightarrow x'' - 2x = 0$$

Then $r^2 - 2 = 0$

$$\Rightarrow r = \pm\sqrt{2} \qquad \text{So } \underline{x = Ae^{\sqrt{2}t} + Be^{-\sqrt{2}t}}$$

Then as $y = x'$

$$\Rightarrow \underline{y = \sqrt{2}Ae^{\sqrt{2}t} - \sqrt{2}Be^{-\sqrt{2}t}}$$



$$x^2 = A^2 e^{\sqrt{2}t} \cdot e^{\sqrt{2}t} + 2AB e^{\sqrt{2}t - \sqrt{2}t} + B^2 e^{-2\sqrt{2}t}$$

We compute

$$x^2 \underline{- \tfrac{1}{2}} y^2$$

$$= A^2 e^{2\sqrt{2}t} + 2AB + B^2 e^{-2\sqrt{2}t}$$

$$\underline{-\tfrac{1}{2}}\left(2A^2 e^{2\sqrt{2}t} - 2\cdot 2AB + 2B^2 e^{-2\sqrt{2}t}\right)$$

$$= 2AB + \tfrac{1}{2}\cdot 4AB = 4AB$$

Thus

$$x^2 - \tfrac{1}{2}y^2 = 4AB$$

$$\Rightarrow \frac{x^2}{4AB} - \frac{y^2}{8AB} = 1$$

which is a eqn of a hyperbala.