3.6 Forced Oscillations and Resonance

In this section, we will talk about the systems with forced oscillations.

We have the differential equation

$$mx'' + cx' + kx = F(t)$$

with

$$F(t) = F_0 \cos \omega t \qquad \text{or} \qquad F(t) = F_0 \sin \omega t$$

where the constant F_0 is the amplitude of the periodic force and ω is its circular frequency.

Undamped Forced Oscillations

We set $c = 0$ and consider

$$mx'' + kx = F_0 \cos \omega t \tag{1}$$

Discussion:

- By Section 3.4, the complementary function is

$$x_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

  where $w_0 = \sqrt{\frac{k}{m}} \Rightarrow k = w_0^2 m.$

- Assume $w_0 \neq w$. we want to find a particular solution x_p of Eq(1).
- Assume $x_p = A \cos \omega t, \quad x_p'' = -A\omega^2 \cos wt$ then

$$mx_p'' + kx_p = -Am\omega^2 \cos wt + kA \cos \omega t = F_0 \cos wt$$

$$\Rightarrow A\left(k - m\omega^2\right) = F_0 \Rightarrow A = \frac{F_0}{k - mw^2} = \frac{F_0/m}{\omega_0^2 - w^2},$$

  the last equation is from the fact that $k = \omega_0^2 m.$

- Thus

$$x_p = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t.$$

- Therefore the general solution

$$x(t) = x_c(t) + x_p(t) = c_1 \cos w_0 t + c_2 \sin \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

$$\Rightarrow \boxed{x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t}$$

- So $x(t)$ is a superposition of two oscillations.

Example 1 Express the solution of the given initial value problem as a sum of two oscillations. Graph the solution function $x(t)$ in such a way that you can identify and label its period.

$$x'' + 25x = 9\cos 2t; \qquad x(0) = 0, \quad x'(0) = 0$$

ANS: • Find x_c. $r^2 + 25 = 0 \Rightarrow r = \pm 5i$

$$x_c = C_1 \cos 5t + C_2 \sin 5t$$

• Find x_p. Assume $x_p = A\cos 2t$, then $x_p'' = -4A\cos 2t$.

$x_p' = -2A\sin 2t$

$$x_p'' + 25x_p = (-4A + 25A)\cos 2t = 9\cos 2t$$

$$\Rightarrow 21A = 9 \Rightarrow A = \frac{3}{7}$$

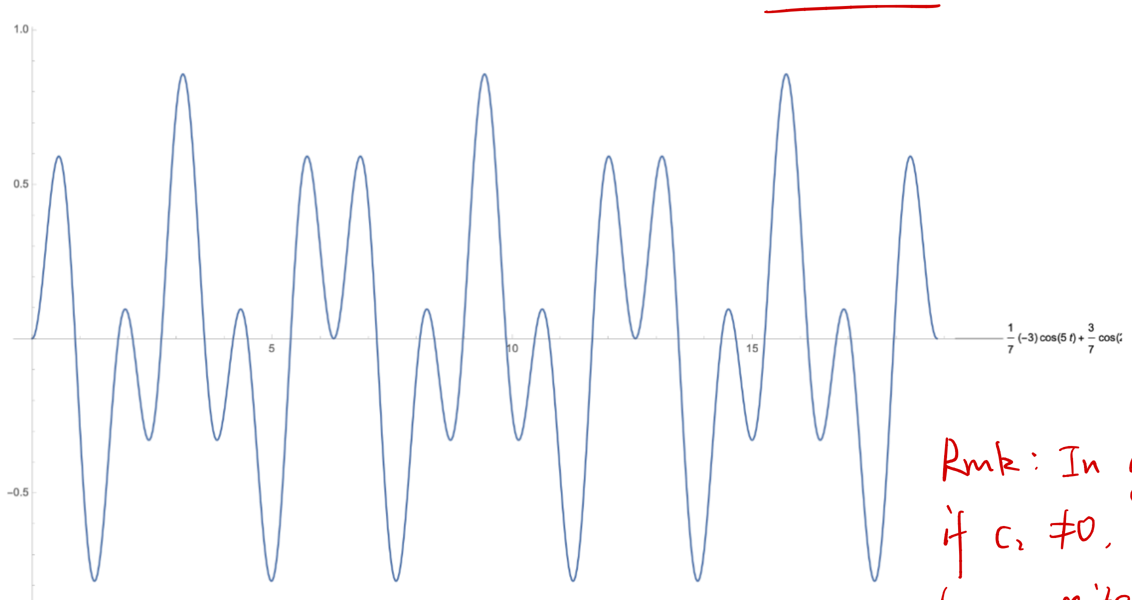
• Use the initial value to find $x(t)$

$$x(t) = x_c + x_p = C_1 \cos 5t + C_2 \sin 5t + \frac{3}{7}\cos 2t$$

As $x(0) = 0$, $x(0) = C_1 + \frac{3}{7} = 0 \Rightarrow C_1 = -\frac{3}{7}$

As $x'(0) = 0$, $x'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t - \frac{6}{7}\sin 2t$

$$x'(0) = 5C_2 = 0 \Rightarrow \underline{C_2 = 0}.$$



Thus $x(t) = -\frac{3}{7}\cos 5t + \frac{3}{7}\cos 2t$.
which is a sum of two oscillations.

Rmk: In geneal, if $C_2 \neq 0$, we need to rewrite x_c as the form $x_c = C\cos(\omega_0 t - \alpha)$

The period T of $x(t)$ is the least common mutiple of the periods of the two oscillations. $\frac{2\pi}{5}$ and $\frac{2\pi}{2}$, which is 2π.

Damped Forced Oscillations

$$mx'' + cx' + kx = F_0 \cos \omega t$$

- **transient solution** $x_{\text{tr}}(t) = x_c(t), \quad x_c(t) \to 0$ as $t \to \infty$.
- **steady periodic solution** $x_{\text{sp}}(t) = x_p(t)$

x_p

Example 2 Find the steady periodic solution $x_{\text{sp}} = C \cos(\omega t - \alpha)$ of the given equation $mx'' + cx' + kx = F(t)$ with periodic forcing function $F(t)$ of frequency ω. Then graph $x_{\text{sp}}(t)$ together with (for comparison) the adjusted forcing function $F_1(t) = F(t)/m\omega$.

$m=1$ $\quad F_0 = 82$

$c=2$

$k=26$

$w=4$

$$x'' + 2x' + 26x = 82 \cos 4t$$

ANS: Assume $x_p = A\cos 4t + B\sin 4t$.

$$x_p' = -4A\sin 4t + 4B\cos 4t$$

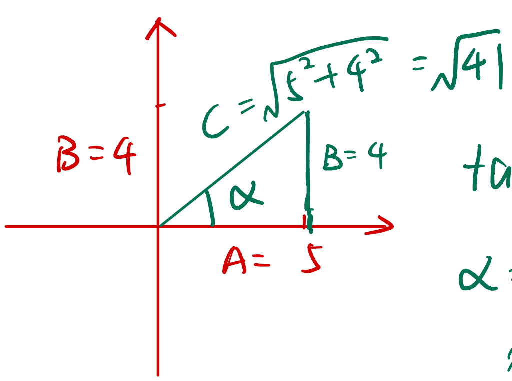
$$x_p'' = -16A\cos 4t - 16B\sin 4t$$

Then
$$\begin{aligned} x_p'' + 2x_p' + 26x_p &= -16A\cos 4t - 16B\sin 4t \\ &\quad + 2(-4A\sin 4t + 4B\cos 4t) \\ &\quad + 26(A\cos 4t + B\sin 4t) \\ &= 82\cos 4t \end{aligned}$$

$$\Rightarrow \begin{cases} 10A + 8B = 82 \\ -8A + 10B = 0 \end{cases} \Rightarrow \begin{cases} A = 5 \\ B = 4. \end{cases}$$

Then $x_{sp} = x_p = \underset{A}{5}\cos 4t + \underset{B}{4}\sin 4t$.

$c\cos(\omega t - \alpha)$

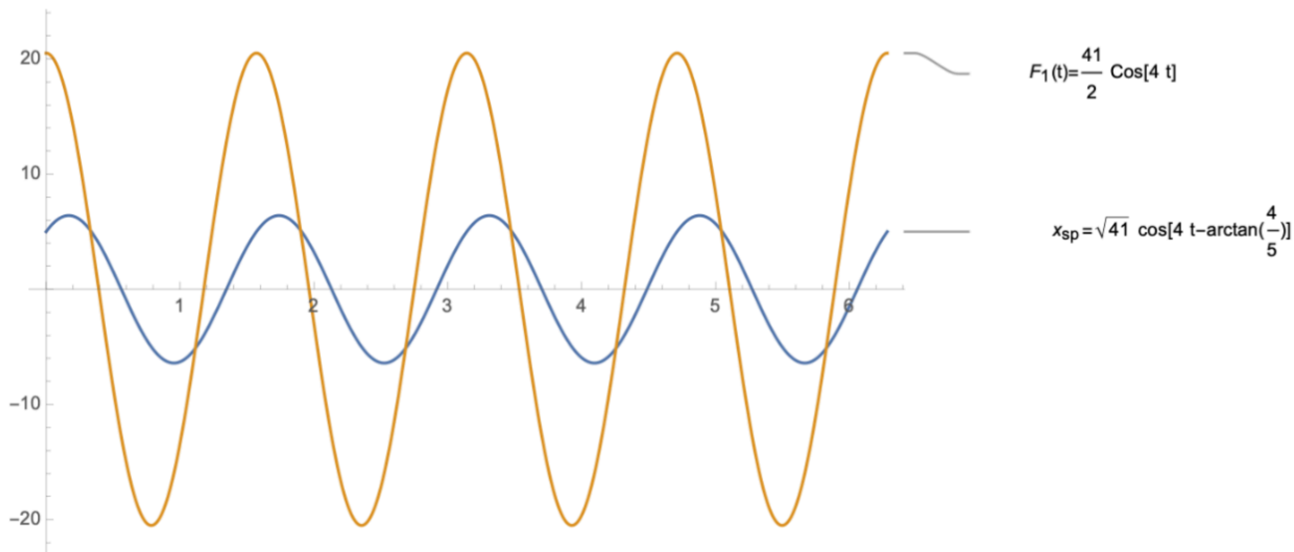


$$\tan\alpha = \frac{4}{5}$$

$$\alpha = \arctan\frac{4}{5} \approx 0.6747$$

Then $\boxed{x_{sp} = \sqrt{41}\cos(4t - 0.6747)}$

$$F_1 = \frac{F(t)}{m\omega} = \frac{82\cos 4t}{1 \cdot 4} = \frac{41}{2}\cos 4t$$



Example 3 Find and plot both the steady periodic solution $x_{\text{sp}} = C\cos(\omega t - \alpha)$ of the given differential equation and the actual solution $x(t) = x_{\text{sp}}(t) + x_{\text{tr}}(t)$ that satisfies the given initial conditions.

$$x'' + 2x' + 26x = 82\cos 4t; \qquad x(0) = 6, \quad x'(0) = 0$$

ANS: By Example, $x_{sp}(t) = x_p = 5\cos 4t + 4\sin 4t$.

Now we find $x_{tr}(t) = x_c(t)$.

$$r^2 + 2r + 26 = 0$$

$$\Rightarrow \quad r = \frac{-2 \pm \sqrt{4 - 4\times 26}}{2} = \frac{-2 \pm \sqrt{-100}}{2}$$

$$= -1 \pm 5i$$

$$x_c(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t)$$

Now $x(t) = x_c(t) + x_p(t)$

$$\Rightarrow x(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t) + 5\cos 4t + 4\sin 4t$$

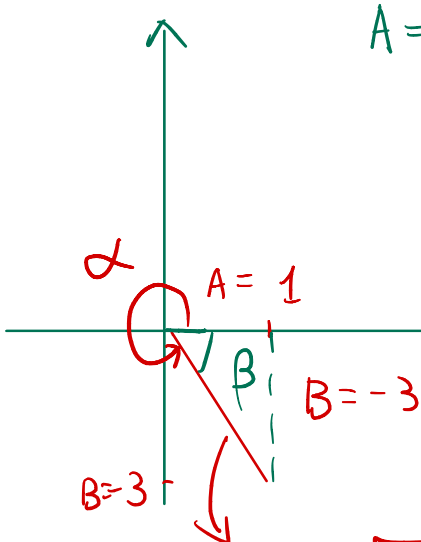
As $x(0)=6$, $x'(0)=0$.

$$c_1 = 1 \quad \text{and} \quad c_2 = -3$$

Thus $$x_c = e^{-t}(\cos 5t - 3\sin 5t)$$

$$Ce^{-t}\cos(5t-\alpha)$$

$A=1$, $B=-3$



$$\alpha = 2\pi - \beta$$

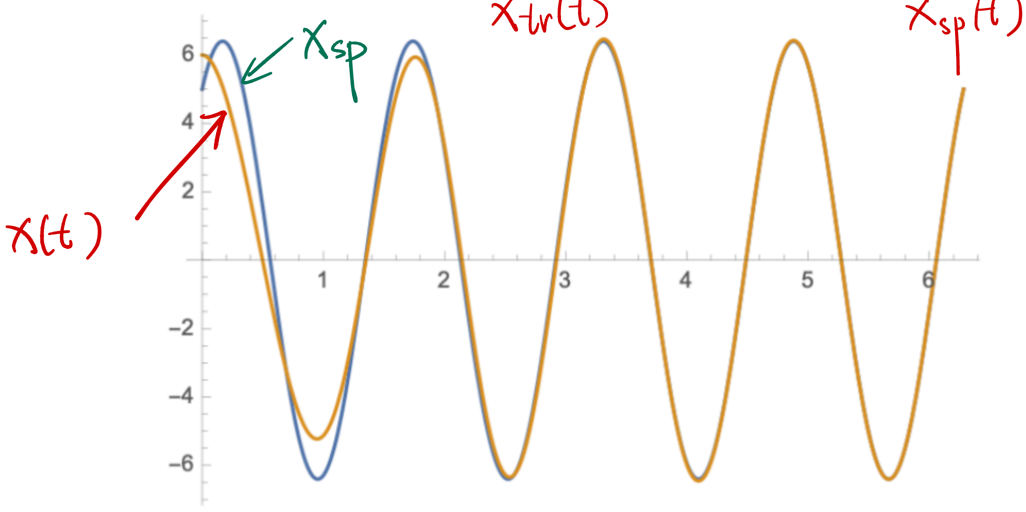
$$\tan\beta = \frac{3}{1} = 3$$

$$\beta = \arctan 3$$

So $\alpha = 2\pi - \arctan 3$

$$C = \sqrt{A^2+B^2} = \sqrt{10} \qquad \approx 5.0341$$

Thus $$x(t) = \underbrace{\sqrt{10}\,e^{-t}\cos(5t-5.0341)}_{x_{tr}(t)} + \underbrace{\sqrt{41}\cos(4t-0.6747)}_{x_{sp}(t)}$$



Example 4 The following question gives the parameters for a forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos \omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady periodic forced oscillations with frequency ω.
Sketch the graph of $C(\omega)$ and find the practical resonance frequency ω (if any).

x_p

(a) $m = 1, c = 2, k = 26, F_0 = 82$

The maximal value of $C(\omega)$ of x_p. This may not exist.

(b) $m = 1, c = 2, k = 2, F_0 = 2$. (exercise)

ANS: We have

$$x'' + 2x' + 26x = 82 \cos wt.$$

Assume $x_p = A \cos wt + B \sin wt$. then

$$x_p' = -Aw \sin wt + Bw \cos wt$$

$$x_p'' = -Aw^2 \cos wt - Bw^2 \sin wt$$

Then $-Aw^2 \cos wt - Bw^2 \sin wt + 2(-Aw \sin wt + Bw \cos wt)$
$+ 26(A \cos wt + B \sin wt) = 82 \cos wt$

$$\begin{cases} A(-w^2+26) + 2B = 82 \\ -2Aw + B(-w^2+26) = 0 \end{cases} \Rightarrow \begin{cases} A(w) = -\dfrac{82(-26+w^2)}{w^4 - 48w^2 + 676} \\ B(w) = \dfrac{164w}{w^4 - 48w^2 + 676} \end{cases}$$

$$\Rightarrow C = \sqrt{A^2 + B^2}$$

$$\Rightarrow C(w) = \frac{82}{\sqrt{w^4 - 48w^2 + 676}}$$

Then we graph the function $C(w)$.

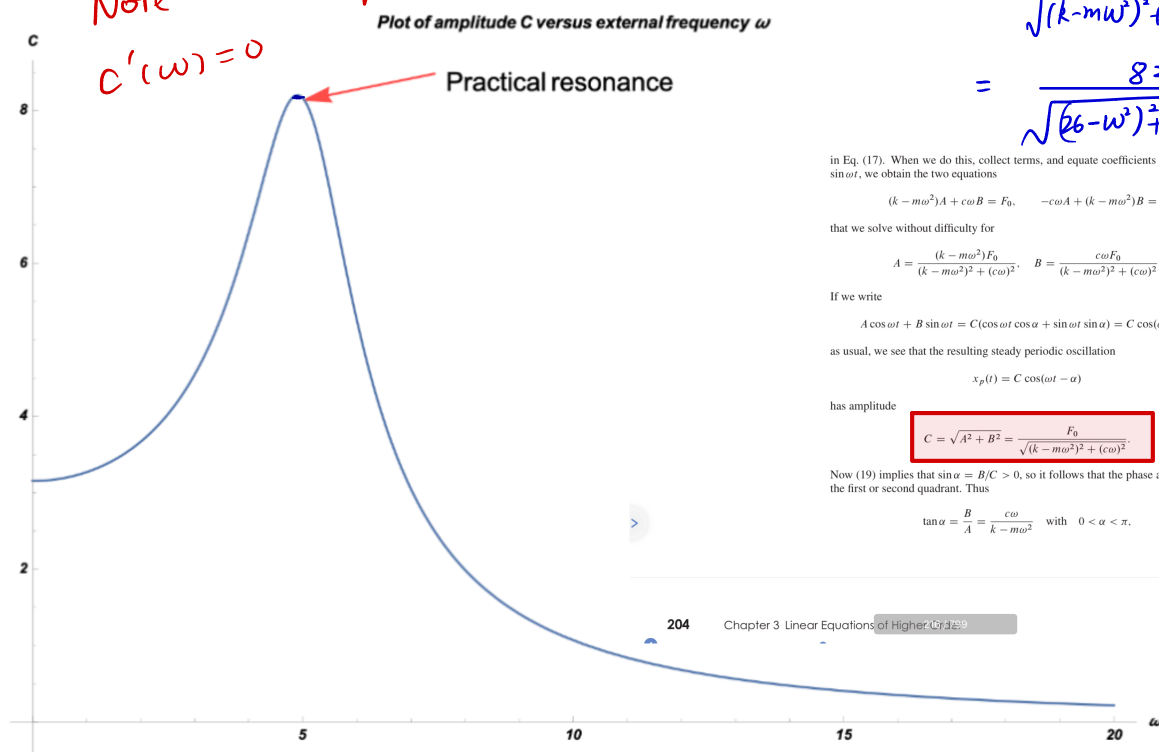
By the graph, we know that the pratical resonce occurs when $C'(w) = 0$. Let $C'(w) = 0 \Rightarrow C'(w) = \dfrac{-164w(w^2-24)}{(w^4 - 48w^2 + 676)^{3/2}} = 0$

$\Rightarrow w = 0$ or $w = \pm\sqrt{24}$. Then $C_{max} = C(\sqrt{24}) = \dfrac{82}{\sqrt{24^2 - 48 \cdot 24 + 676}} = \cdots$

You can also use the formula (21) directly from the book. So

$$C(\omega) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} = \frac{82}{\sqrt{(26-\omega^2)^2 + (2\omega)^2}}$$

Note at this pt. $C'(\omega) = 0$



in Eq. (17). When we do this, collect terms, and equate coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain the two equations

$$(k - m\omega^2)A + c\omega B = F_0, \qquad -c\omega A + (k - m\omega^2)B = 0 \qquad (18)$$

that we solve without difficulty for

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}. \qquad (19)$$

If we write

$$A\cos\omega t + B\sin\omega t = C(\cos\omega t\cos\alpha + \sin\omega t\sin\alpha) = C\cos(\omega t - \alpha)$$

as usual, we see that the resulting steady periodic oscillation

$$x_p(t) = C\cos(\omega t - \alpha) \qquad (20)$$

has amplitude

$$\boxed{C = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.} \qquad (21)$$

Now (19) implies that $\sin\alpha = B/C > 0$, so it follows that the phase angle α lies in the first or second quadrant. Thus

$$\tan\alpha = \frac{B}{A} = \frac{c\omega}{k - m\omega^2} \quad \text{with} \quad 0 < \alpha < \pi, \qquad (22)$$

(b) $m = 1, c = 2, k = 2, F_0 = 2$. (exercise)

We have $x'' + 2x' + 2x = 2\cos\omega t$

Assume $x_p = A\cos\omega t + B\sin\omega t$

Then

$$x_p'' + 2x_p' + 2x_p = 2\cos\omega t$$

$$\Rightarrow \begin{cases} (2-\omega^2)A + 2\omega B = 2 \\ -2\omega A + (2-\omega^2)B = 0 \end{cases} \Rightarrow \begin{cases} A = \dfrac{2(2-\omega^2)}{4+\omega^4} \\ B = \dfrac{4\omega}{4+\omega^4} \end{cases}$$

Then $C(\omega) = \sqrt{A^2 + B^2} = \dfrac{2}{\sqrt{4+\omega^4}}$ with $C(0) = 1$.

$C(\omega)$ is a decreasing function.

Thus there is no pratical resonace