3.5 Nonhomogeneous Equations and Undetermined Coefficients

Consider the general nonhomogeneous nth-order linear equation with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \tag{1}$$

A general solution of Eq.(1) has the form

$$\boxed{y = y_c + y_p}$$

where the complementary function $y_c(x)$ is a general solution of the associated homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

and $y_p(x)$ is a particular solution of Eq. (1).

Method of Undetermined Coefficients

Example 1 Find a general solution y of the given equation. ($f(x)$ is a polynomial.)

$$y'' + 4y = 3x^2 \qquad \otimes$$

ANS: We have $y = y_c + y_p$. where y_c is a general solution to $y'' + 4y = 0$. and y_p is a particular solution to $\otimes$.

- Find y_c. The char. eqn. is $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

- Find y_p. Guess $y_p = Ax^2 + Bx + C$.

$$y_p' = 2Ax + B$$

$$y_p'' = 2A.$$

We have $y_p'' + 4y_p = 3x^2$

$$\Rightarrow 2A + 4(Ax^2 + Bx + C) = 3x^2$$

$$\Rightarrow 4Ax^2 + 4Bx + (2A + 4C) = 3x^2$$

$$\Rightarrow \begin{cases} 4A = 3 \\ B = 0 \\ 2A + 4C = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{3}{4} \\ B = 0 \\ C = -\frac{3}{8}. \end{cases} \qquad y_p = \frac{3}{4}x^2 - \frac{3}{8}.$$

Thus the general solution is

$$y(x) = y_c + y_p$$

$$= C_1 \cos 2x + C_2 \sin 2x + \frac{3}{4}x^2 - \frac{3}{8}.$$

Exercise 2 Find a particular solution y_p of the given equation. ($f(x)$ is an exponential fuction e^{rx}.)

$$y'' - 3y' - 4y = 3e^{2x}$$

Hint: Try $y_p(x) = Ae^{2x}$ and solve for A. $\Rightarrow$ $A = -\frac{1}{2}$

ANS: We guess $y_p(x) = Ae^{2x}$. Then $y_p' = 2Ae^{2x}$, $y_p'' = 4Ae^{2x}$

Then $y_p'' - 3y_p' - 4y_p = 3e^{2x}$

$\Rightarrow \cancel{4Ae^{2x}} - 6Ae^{2x} - \cancel{4Ae^{2x}} = 3e^{2x}$

$\Rightarrow -6A = 3 \Rightarrow A = -\frac{1}{2}$.

Thus $y_p = -\frac{1}{2}e^{2x}$

Example 3 Find a particular solution y_p of the given equation. ($f(x)$ is $\cos kx$ or $\sin kx$.)

$$y'' - 3y' - 4y = 2\sin x$$

ANS: We try $y_p = A\sin x + B\cos x$

$y_p' = A\cos x - B\sin x$

$y_p'' = -A\sin x - B\cos x$

Then $y_p'' - 3y_p' - 4y_p = (-A\sin x - B\cos x) - 3(A\cos x - B\sin x)$

$- 4(A\sin x + B\cos x)$

$= (-A + 3B - 4A)\sin x + (-B - 3A - 4B)\cos x$

$= 2\sin x$

$\Rightarrow \begin{cases} -5A + 3B = 2 \\ -5B - 3A = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{5}{17} \\ B = \frac{3}{17}. \end{cases}$ Thus

$$\boxed{y_p = -\frac{5}{17}\sin x + \frac{3}{17}\cos x}$$

Example 4 Find a particular solution y_p of the given equation. ($f(x)$ is $e^{rx}\cos kx$ or $e^{rx}\sin kx$)

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

ANS: We try $y_p = Ae^x\cos 2x + Be^x \sin 2x$.

$$y_p' = A(e^x\cos 2x - 2e^x\sin 2x) + B(e^x\sin 2x + 2e^x\cos 2x)$$
$$= (A+2B)e^x\cos 2x + (-2A+B)e^x\sin 2x$$
$$y_p'' = (-3A+4B)e^x\cos 2x + (-4A-3B)e^x\sin 2x$$

Now we have

$$-8e^x\cos 2x = y_p'' - 3y_p' - 4y_p$$
$$= (-3A+4B)e^x\cos 2x + (-4A-3B)e^x\sin 2x$$
$$-3\left[(A+2B)e^x\cos 2x + (-2A+B)e^x\sin 2x\right]$$
$$-4\left[Ae^x\cos 2x + Be^x\sin 2x\right]$$

$$\begin{cases}10A+2B=8\\ 2A-10B=0\end{cases} \Rightarrow \begin{cases}A=10/13\\ B=2/13.\end{cases}$$ Thus $y_p = \frac{10}{13}e^x\cos 2x + \frac{2}{13}e^x\sin 2x$

Example 5 Find a particular solution y_p of the given equation. ($f(x)$ is a combination)

$$y'' - 3y' - 4y = \underbrace{3e^{2x}}_{f_1(x)} + \underbrace{2\sin x}_{f_2(x)} \underbrace{- 8e^x\cos 2x}_{f_3(x)}$$

ANS: We can split the eqn into 3 eqns.

$$y'' - 3y' - 4y = 3e^{2x} = f_1(x)$$
$$y'' - 3y' - 4y = 2\sin x = f_2(x)$$
$$y'' - 3y' - 4y = -8e^x\cos 2x = f_3(x)$$

Then by Ex. 2-4. we have

$$y_p = -\frac{1}{2}e^{2x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x + \frac{10}{13}e^x\cos 2x + \frac{2}{13}e^x\sin 2x$$

This is the case that f(x) is a solution to the corresponding homogeneous eqn.

The Case of Duplication

Example 6 Find a particular solution of $y'' - 4y = 2e^{2x}$ $= f(x)$

ANS: If we try $\cancel{y_p = Ae^{2x}}$, then $y_p'' = 4Ae^{2x}$

Then $y_p'' - 4y_p = 4Ae^{2x} - 4Ae^{2x} = 0 \neq 2e^{2x}$

The reason is

The char. eqn for the homogeneous part $y'' - 4y = 0$. is

$$r^2 - 4 = 0 \Rightarrow r = \pm 2.$$

Then Ae^{2x} is a solution to $y'' - 4y = 0$.

We try $y_p = Axe^{2x}$, then $y_p' = A(e^{2x} + 2xe^{2x})$

$$y_p'' = 2Ae^{2x} + 2A(e^{2x} + 2xe^{2x}) = 4Ae^{2x} + 4Axe^{2x}$$

Then $y_p'' - 4y_p = 4Ae^{2x} + \cancel{4Axe^{2x}} - \cancel{4Axe^{2x}} = 2e^{2x}$

$\Rightarrow 4A = 2 \Rightarrow A = \frac{1}{2}$.

Thus $\boxed{y_p = \frac{1}{2}xe^{2x}}$

In general, we have the following:

Polynomial

If the function $f(x)$ is of either form of $P_m(x)e^{rx}\cos kx$, $P_m(x)e^{rx}\sin kx$, we can assume

$$y_p(x) = x^s[(A_0 + A_1x + \cdots + A_mx^m)e^{rx}\cos kx + (B_0 + B_1x + \cdots + B_mx^m)e^{rx}\sin kx]$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function y_c.

Summary (Reading)

We summarize the steps of finding the solution of an initial value problem consisting of a nonhomogeneous equation of the form

$$ay'' + by' + cy = f(x) \qquad (2)$$

where a, b, c are constants, together with a given set of initial conditions:

1. Find the general solution of the corresponding homogeneous equation. $ay'' + by' + cy = 0$
2. Make sure that function $f(x)$ in Eq. (2) belongs to the class of functions discussed above; that is, be sure it involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of such functions. If this is not the case, use the method of variation of parameters (discussed in the following part of this section).
3. If $f(x) = f_1(x) + \cdots + f_n(x)$, that is, if $f(x)$ is a sum of n terms, then form n subproblems, each of which contains only one of the terms $f_1(x), \ldots, f_n(x)$. The ith subproblem consists of the equation
$$ay'' + by' + cy = f_i(x)$$
where i runs from 1 to n. (see Example 5)
4. For the ith subproblem assume a particular solution $y_{ip}(x)$ consisting of the appropriate exponential function, sine, cosine, polynomial, or combination thereof. If there is any duplication in the assumed form of $y_{ip}(x)$ with the solutions of the homogeneous equation (found in step 1), then multiply $y_{ip}(x)$ by x, or (if necessary) by x^2, so as to remove the duplication. (See the table for general cases)
5. Find a particular solution $y_{ip}(t)$ for each of the subproblems. Then the sum $y_{1p}(t) + y_{2p}(t) + \cdots + y_{np}(t)$ is a particular solution of the full nonhomogeneous Eq (2).
6. Form the sum of the general solution of the homogeneous equation (step 1) and the particular solution of the nonhomogeneous equation (step 5). This is the general solution of the nonhomogeneous equation.
7. Use the initial conditions to determine the values of the arbitrary constants remaining in the general solution.

$f(x)$	y_p
$P_m = b_0 + b_2 x + \cdots + b_m x^m$	$x^s(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1 x + x_2 x^2 + \cdots + A_m x^m)\cos kx +$ $(B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)\sin kx]$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\sinh x = \dfrac{e^x - e^{-x}}{2}$

Example 7 Set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.

(a) $y'' - 4y = \sinh 2x = \dfrac{e^{2x} - e^{-2x}}{2} = \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x} = f_1(x) + f_2(x)$

ANS: Notice that the char. eqn. for the corresponding homogeneous eqn is $r^2 - 4 = 0 \Rightarrow r = \pm 2$. Thus

$$y_c = C_1 e^{2x} + C_2 e^{-2x}$$

Thus

$$y_p = A\underset{\substack{\uparrow \\ \text{since } e^{2x} \\ \text{appears once in} \\ y_c}}{x} e^{2x} + B\underset{\substack{\uparrow \\ \text{since } e^{-2x} \\ \text{appears once in } y_c}}{x} e^{-2x}$$

polynomials

(b) $y'' + 3y' + 2y = x(e^{-x} - e^{-2x}) = \underline{x}e^{-x} - \underline{x}e^{-2x} = f_1(x) + f_2(x)$

ANS: The char. eqn. for the homogenous part is

$$r^2 + 3r + 2 = 0$$
$$\Rightarrow (r+1)(r+2) = 0$$
$$\Rightarrow r = -1 \text{ or } r = -2$$

Thus $y_c = C_1 e^{-x} + C_2 e^{-2x}$

Thus we assume

$$y_p = \overbrace{x\,\underline{(Ax+B)}\,e^{-x}}^{\text{corresponds to } f_1(x) = xe^{-x}} + \overbrace{x\,\underline{(Cx+D)}\,e^{-2x}}^{\text{corresponds to } f_2(x) = -xe^{-2x}}$$

since e^{-x} appears once in y_c

from $\underline{x}e^{-x}$

since e^{-2x} appears once in y_c

from $\underline{x}e^{-2x}$

(c) (Exercise) $(D-2)^3(D^2+9)y = x^2e^{2x} + x\sin 3x = f_1(x) + f_2(x)$

diff.

ANS: The char. eqn. is

$$(r-2)^3(r^2+9)=0$$

$$\Rightarrow r = 2, 2, 2, 3i, -3i.$$

Then $y_c = (C_1 + C_2x + C_3x^2)e^{2x} + C_4\cos 3x + C_5\sin 3x$

· Consider $f_1(x) = x^2e^{2x}$. We assume

$$y_{p1}(x) = x^3(Ax^2+Bx+C)e^{2x}$$ from x^2e^{2x}

since e^{2x} appears 3 times in y_c.

· Consider $f_2(x) = x\sin 3x$. We assume

$$y_{p_2}(x) = x(Dx+E)\cos 3x + x(Fx+G)\sin 3x$$ from $x\sin 3x$

since $\sin 3x$ appears

Thus $\boxed{y_p(x) = x^3(Ax^2+Bx+C)e^{2x} + x(Dx+E)\cos 3x + x(Fx+G)\sin 3x}$

(d) (Similar to online HW 18. Q9. Note it is a case of duplication)

$$y^{(5)} + 2y^{(3)} + 2y'' = 3x^2 - 1$$

$3x^2-1$ is a solution to $y^{(5)} + 2y^{(3)} + 2y'' = 0$

Try $y_p = Ax^2 + Bx + C$ and plug into the eqn. This won't work.

Note this is the case when we have

1. $r = 0$ for the characteristic equation.
2. $f(x)$ is a polynomial $P_m(x)$.

If we have $r = 0$ from the characteristic equation, recall the table with $r = 0$:

$f(x)$ 1	y_p 1
$P_m(x)\cancel{e^{rx}}$ $r=0$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)\cancel{e^{rx}}$ $r=0$

ANS: The char. eqn for the homogeneous part is

$$r^5 + 2r^3 + 2r^2 = 0$$

$$\Rightarrow r^2(r^3 + 2r + 2) = 0$$

$$\Rightarrow r = 0 \text{ (twice)}$$

The part in y_c corresponds to $r=0$ is $(C_1 + C_2x)e^{0x} = C_1 + C_2x$

This means any function of the form $C_1 + C_2x$ will to a solution for the homogeneous part.

So we assume

$$y_p = x^2(Ax^2 + Bx + C)$$

proof of this

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Variation of Parameters

THEOREM 1 Variation of Parameters

If the nonhomogeneous equation $y'' + P(x)y' + Q(x)y = f(x)$ has complementary function $y_c(x) = c_1y_1(x) + c_2y_2(x)$, then a particular solution is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation.

If we write $u_1 = -\int \frac{y_2(x) f(x)}{W(x)} dx$, $u_2 = \int \frac{y_1(x) f(x)}{W(x)} dx$.

Then $y_p = u_1 y_1 + u_2 y_2$.

Example 8 Use the method of variation of parameters to find a particular solution of the tiven differential equation.

$$y'' + 9y = 2\sec 3x = 2\cdot\frac{1}{\cos 3x}$$

ANS: First, let's find y_1 and y_2.

The char. eqn is $r^2+9=0 \Rightarrow r = \pm 3i$.

Then $y_c = c_1 y_1 + c_2 y_2 = c_1 \underbrace{\cos 3x}_{y_1} + c_2 \underbrace{\sin 3x}_{y_2}$

Then

$$W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$$

We have

$$u_1(x) = -\int \frac{y_2(x) f(x)}{W(x)} dx = -\int \frac{\sin 3x \cdot 2 \cdot \frac{1}{\cos 3x}}{3} dx = -\frac{2}{3}\int \tan 3x \, dx$$

$$= -\frac{2}{9}\int \tan 3x \, d3x = \frac{2}{9} \ln|\cos 3x|$$

$$\int \tan t \, dt = -\ln|\cos t| dt + C$$

$$u_2(x) = \int \frac{y_1(x) f(x)}{W(x)} dx = \int \frac{\cos 3x \cdot 2 \cdot \frac{1}{\cos 3x}}{3} dx = \frac{2}{3}\int dx = \frac{2}{3}x$$

So $\boxed{y_p = u_1 y_1 + u_2 y_2 = \frac{2}{9}\cos 3x \ln|\cos 3x| + \frac{2}{3}x \cdot \sin 3x}$

$$y(x) = y_c + y_p$$

Non-homogeneous Euler Equation

Example 9 In the following question, a nonhomogeneous second-order linear equation and a complementary function y_c are given. Find a particular solution y_p of the equation.

$$x^2y'' - 3xy' + 4y = x^4; y_c = x^2(c_1 + c_2 \ln x) = c_1 \underbrace{x^2}_{y_1} + c_2 \underbrace{x^2 \ln x}_{y_2}$$

ANS: Standard form : $y'' - \frac{3}{x} y' + \frac{4}{x^2} y = x^2 = f(x)$

$$y_c = x^2 (c_1 + c_2 \ln x) = c_1 x^2 + c_2 x^2 \ln x = c_1 y_1 + c_2 y_2$$

Then

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x\ln x + x \end{vmatrix} = \cancel{2x^3 \ln x} + x^3 - \cancel{2x^3 \ln x} = x^3$$

Then

$$u_1(x) = -\int \frac{y_2(x) \cdot f(x)}{W(x)} dx = -\int \frac{(x^2 \ln x)\ x^2}{x^3} dx$$

$$\int u\,dv = uv - \int v\,du$$

$$= -\int x \underbrace{\ln x}_{u} dx$$

$$= -\frac{1}{2} \int \ln x \, d x^2$$

$$= -\frac{1}{2} \left[x^2 \ln x - \int x^2 d \ln x \right]$$

$$= -\frac{1}{2} \left[x^2 \ln x - \int x \, dx \right]$$

$$= -\frac{1}{2} \left[x^2 \ln x - \frac{1}{2} x^2 \right]$$

$$= -\frac{1}{2} x^2 \ln x + \frac{1}{4} x^2$$

$$u_2(x) = \int \frac{y_1(x) f(x)}{W(x)} dx = \int \frac{x^2 \cdot x^2}{x^3} dx = \int x\,dx$$

$$= \frac{1}{2} x^2$$

Thus

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \underline{\left(-\frac{1}{2}x^2\ln x + \frac{1}{4}x^2\right)} \cdot \underline{x^2} + \underline{\frac{1}{2}x^2 \cdot x^2 \ln x}$$

$$\Rightarrow \quad y_p = \frac{1}{4}x^4$$