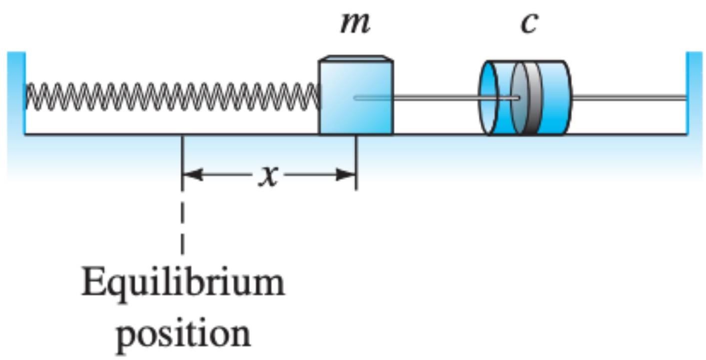
3.4 Mechanical Vibrations

Mass-spring-dashpot system



- Restorative force $F_S = -kx$, where $k > 0$ is **spring constant** (Hooke's law).
- The dashpot provides force $F_R = -cv = -c\dfrac{dx}{dt}$, where $c > 0$ is **damping constant**.
- **External force** $F_E = F(t)$.
- The total force acting of the mass is $F = F_S + F_R + F_E$.
- Using Newton's law,

$$F = ma = m\frac{d^2x}{dt^2} = mx''$$

  we have the following second-order linear differential equation

$$mx'' + cx' + kx = F(t)$$

- If $c = 0$, we call the motion **undamped**. If $c > 0$, we call the motion **damped**.
- If $F(t) = 0$, we call the motion **free**. If $F(t) \neq 0$, we call the motion **forced**.

🤓 **An important note before we start analyzing the general cases:**

Rather than memorizing the various formulas given in the discussion below, it is better to practice a particular case to set up the differential equation and then solve it directly.

1. Free Undamped Motion ($c = 0$ and $F(t) = 0$)

Our general differential equation takes the simpler form

$$mx'' + kx = 0 \Rightarrow x'' + \left(\sqrt{\frac{k}{m}}\right)^2 x = 0$$

- It is convenient to define

$$\omega_0 = \sqrt{\frac{k}{m}}$$

- Then we can rewrite our equation in the form

$$x'' + \omega_0^2 x = 0$$

- Then the characteristic equation is

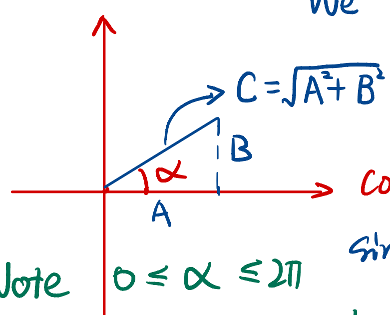
$$r^2 + w_0^2 = 0 \Rightarrow r^2 = -w_0^2 \Rightarrow r = \pm\omega_0 i \text{ (complex conjugate)}$$

- The general solution of this equation is

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

We write $x(t) = C\left(\frac{A}{C}\cos W_0 t + \frac{B}{C}\sin W_0 t\right)$

$= C\left(\cos\alpha\cos W_0 t + \sin\alpha \sin W_0 t\right)$



$\cos\alpha = \frac{A}{C}$

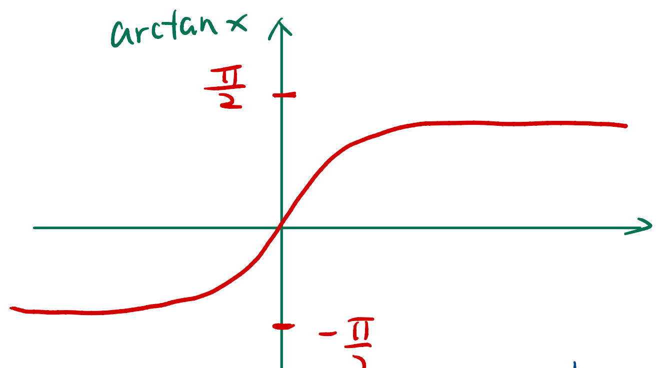
$\sin\alpha = \frac{B}{C}$

$\tan\alpha = \frac{B}{A}$

Note $0 \le \alpha \le 2\pi$

Recall $\cos(a-b) = \cos a\cos b + \sin a \sin b$

Then $x(t) = C\cos(W_0 t - \alpha)$



- Question 1. What is the values of C?

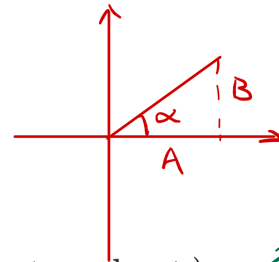
$C = \sqrt{A^2 + B^2}$

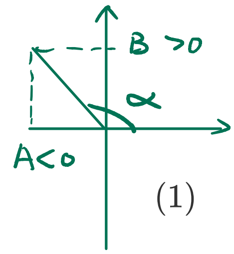
- Question 2: What is the angle α?

· Although $\tan\alpha = \frac{B}{A}$, the angle α is not given by the principal branch of the inverse tangent function, which gives value only in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

· Instead, α is the angle between 0 and 2π such that $\sin\alpha = \frac{B}{C}$, $\cos\alpha = \frac{A}{C}$, where either A or B or both may be negative .

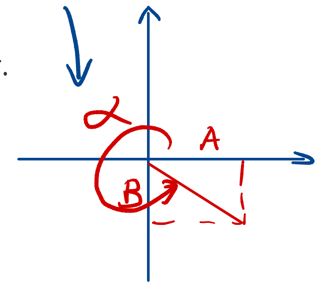
- Thus





$$\alpha = \begin{cases} \tan^{-1}(B/A) & \text{if } A > 0, B > 0 (\text{ first quadrant }) \\ \pi + \tan^{-1}(B/A) & \text{if } A < 0 (\text{ second or third quadrant }) \\ 2\pi + \tan^{-1}(B/A) & \text{if } A > 0, B < 0 \text{ (fourth quadrant)} \end{cases} \quad (1)$$

where $\tan^{-1}(\frac{b}{a})$ is the angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$ given by a calculator or computer.

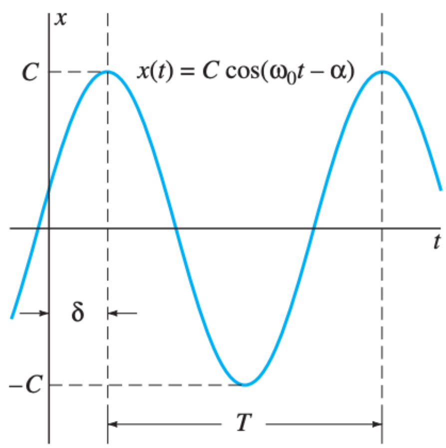


- So we have

$$x(t) = C\cos(\omega_0 t - \alpha)$$

where ω, C and α are obtained as above.

- We call such motion **simple harmonic motion.** A typical graph of such motion is as



- To summarize , it has

Name	**Symbol**	**Quick note**
Amplitude	C	$C = \sqrt{A^2 + B^2}$, where $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$ is the solution for the equation $x'' + \omega_0^2 x = 0$.
Circular frequency	ω_0	$\omega_0 = \sqrt{\frac{k}{m}}$
Phase angle	α	Obtained by formula (1) above
Period	$T = \frac{2\pi}{\omega_0}$	Time required for the system to complete one full oscillation
Frequency	$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$ (In Hz)	It measures the number of complete cycles per second.

Example 1 $\quad \overset{?}{m}x'' + \overset{0}{\cancel{cx'}} + \overset{?}{k}x = \cancel{F(t)}\ 0$

- A body with mass $m = 0.5$ kilogram (kg) is attached to the end of a spring that is stretched 2 meters (m) by a force of 100 newtons (N).
- It is set in motion with initial position $x_0 = 1$ (m) and initial velocity $v = -5$ (m/s). (Note that these initial conditions indicate that the body is displaced to the right and is moving to the left at time $t = 0$.)
- Find the position function of the body in the form $C\cos(\omega_0 t - \alpha)$ as well as the amplitude, frequency and period of its motion.

ANS: Since $F_s = 100 = k \cdot 2 \Rightarrow k = 50$ N/m

Then we have

$$0.5x'' + 50x = 0$$

$$\Rightarrow x'' + 100x = 0 \quad (x'' + \omega_0^2 x = 0)$$

From our previous discussion, we have

$$\omega_0^2 = 100 \Rightarrow \omega_0 = 10 \text{ rad/s}.$$

Thus it will osillate with period $T = \frac{2\pi}{\omega_0} = \frac{\pi}{5}$.

With frequency $\nu = \frac{1}{T} = \frac{5}{\pi} \approx 1.5915$ Hz.

The char. eq $r^2 + 100 = 0 \Rightarrow r = \pm 10i$.

Thus we have $x(t) = A\cos 10t + B\sin 10t$. where A and B are constants.

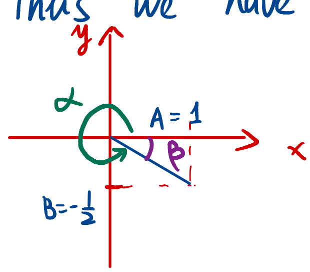
As $x(0) = 1$, $A = 1$. $\quad x'(t) = -10A\sin 10t + 10B\cos 10t$

$x'(0) = 10B = -5$, $\Rightarrow B = -\frac{1}{2}$.

Then $\quad x(t) = \cos 10t - \frac{1}{2}\sin 10t$

The amplitude of the motion is $C = \sqrt{A^2 + B^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$ m.

Thus we have $\quad x(t) = \frac{1}{2}\sqrt{5}\left(\underset{\cos\alpha}{\frac{2}{\sqrt{5}}}\cos 10t - \underset{\sin\alpha}{\frac{1}{\sqrt{5}}}\sin 10t\right)$



By the graph, we have

assume $0 \le \beta \le \frac{\pi}{2}$

$$\alpha = 2\pi - \beta = 2\pi - \tan^{-1}\frac{\frac{1}{2}}{1} = 2\pi - \tan^{-1}\frac{1}{2}$$

$$\approx 5.8195 \text{ rad}.$$

$$\alpha = \begin{cases} \tan^{-1}(B/A) & \text{if } A > 0, B > 0 (\text{ first quadrant }) \\ \pi + \tan^{-1}(B/A) & \text{if } A < 0 (\text{ second or third quadrant }) \\ \underline{2\pi + \tan^{-1}(B/A)} & \text{if } A > 0, B < 0 \ \underline{(\text{fourth quadrant})} \end{cases}$$

We can also use formula 1 . with $A = 1 > 0$, $B = -\frac{1}{2} < 0$

$$\alpha = 2\pi + \tan^{-1}\frac{B}{A}$$

$$= 2\pi + \tan^{-1}\frac{-\frac{1}{2}}{1}$$

$$= 2\pi + \tan^{-1}(-\tfrac{1}{2})$$

$$\approx 5.8195 \text{ rad}$$

Thus $x(t) = \frac{\sqrt{5}}{2}\cos(10t - 5.8195)$

2. Free Damped Motion ($c > 0$ and $F(t) = 0$)

In this case, we consider

$$mx'' + cx' + kx = 0$$

Let $\omega_0 = \sqrt{k/m}$ and $p = \dfrac{c}{2m} > 0$. We have

$$x'' + 2px' + \omega_0^2 x = 0$$

The characteristic equation

$$r^2 + 2pr + \omega_0^2 = 0$$

has roots

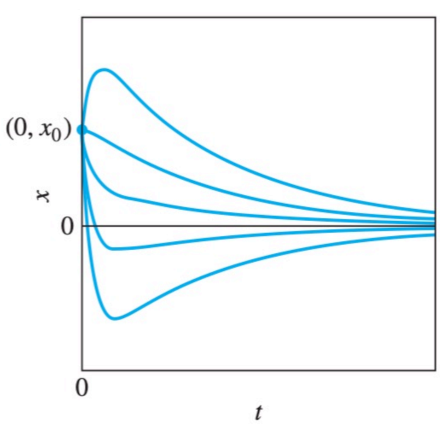
$$r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2} \tag{2}$$

Note

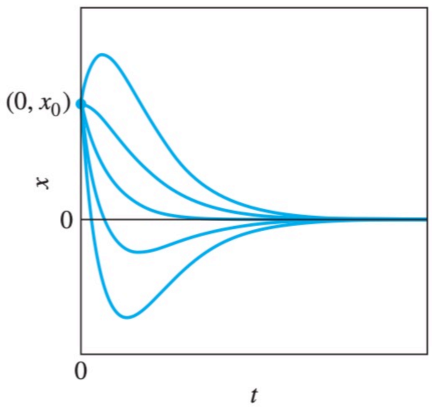
$$p^2 - \omega_0^2 = \frac{c^2 - 4km}{4m^2}$$

We have the following three cases.

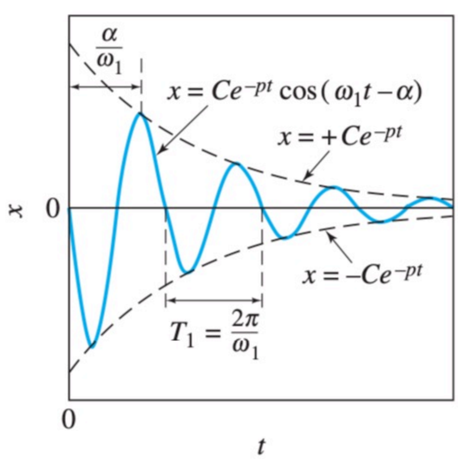
Case 1. Overdamped ($c^2 > 4km$, two distinct real roots)

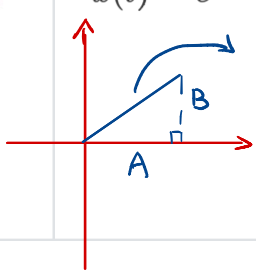
Figure	Analysis
 **FIGURE 3.4.7.** Overdamped motion: $x(t) = c_1e^{r_1t} + c_2e^{r_2t}$ with $r_1 < 0$ and $r_2 < 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.	Eq(3) gives two distinct real roots r_1 and r_2(both < 0). The position function $$x(t) = c_1e^{r_1t} + c_2e^{r_2t}$$ Note $$\lim_{t\to\infty} x(t) = 0$$ (The object will go to the equilibrum position without any ocillations.)

Case 2. Critically damped ($c^2 = 4km$, repeated real roots)

Figure	Analysis
 **FIGURE 3.4.8.** Critically damped motion: $x(t) = (c_1 + c_2 t)e^{-pt}$ with $p > 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.	Eq(3) has roots $r_1 = r_2 = -p$. The general solution for the position function. $$x(t) = e^{-pt}(c_1 + c_2 t)$$ and $$\lim_{t\to\infty} x(t) = 0$$ The resistance of the dashpot is just enough to damp out any oscillations.

Case 3. Underdamped ($c^2 < 4km$, two complex roots)

Figure	Analysis
 **FIGURE 3.4.9.** Underdamped oscillations: $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$.	Eq(3) has roots $r_{1,2} = -p \pm i\sqrt{\omega_0{}^2 - p^2} = -p \pm \omega_1 t$, where $w_1 = \sqrt{\omega_0^2 - p^2}$ The general solution for the position function $x(t) = e^{-pt}(A\cos w_1 t + B\sin \omega_1 t)$



$C = \sqrt{A^2 + B^2}$

$x(t) = e^{-pt} C\left(\frac{A}{C}\cos w_1 t + \frac{B}{C}\sin w_1 t\right)$

$= Ce^{-pt}(\cos\alpha \cos w_1 t + \sin\alpha \sin w_1 t)$

$\Rightarrow x(t) = Ce^{-pt}\cos(w_1 t - \alpha)$

Example 2 Suppose that the mass in a mass–spring–dashpot system with $m = 6, c = 7$, and $k = 2$ is set in motion with $x(0) = 0$ and $x(0)' = 2$.

(a) Find the position function $x(t)$.

(b) Find how far the mass moves to the right before starting back toward the origin.

ANS: We have $6x'' + 7x' + 2x = 0$, $x(0) = 0$, $x'(0) = 2$.

The char. eqn $6r^2 + 7r + 2 = 0$.

$$\Rightarrow r_{1,2} = \frac{-7 \pm \sqrt{49-48}}{12} = \frac{-7 \pm 1}{12} = -\frac{2}{3}, -\frac{1}{2}$$

Thus $x(t) = C_1 e^{-\frac{2}{3}t} + C_2 e^{-\frac{1}{2}t}$

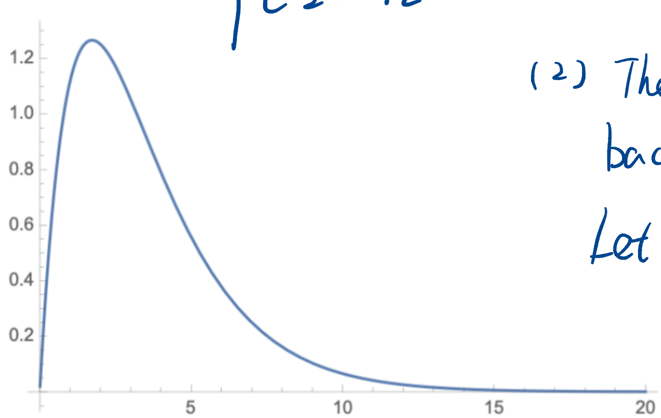
$x(0) = 0$, $C_1 + C_2 = 0$

Since $x'(0) = 2$. $x'(t) = -\frac{2}{3} C_1 e^{-\frac{2}{3}t} - \frac{1}{2} C_2 e^{-\frac{1}{2}t}$

$x'(0) = -\frac{2}{3} C_1 - \frac{1}{2} C_2 = 2$

$$\Rightarrow \begin{cases} C_1 = -12 \\ C_2 = 12. \end{cases}$$

Thus $x(t) = -12 e^{-\frac{2}{3}t} + 12 e^{-\frac{1}{2}t}$



(2) The mass starts to move back when $x'(t) = 0$.

Let $x'(t) = 8 e^{-\frac{2}{3}t} - 6 e^{-\frac{t}{2}} = 0$

$\Rightarrow t = 6 \ln \frac{4}{3} \approx 1.72609s.$

Thus we have

$x(6 \ln \frac{4}{3}) = \frac{81}{64} \approx 1.26563$ m

Example 3 In the following problems **(a)** and **(b)**, a mass m is attached to both a spring (with given spring constant k) and a dash- pot (with given damping constant c). The mass is set in motion with initial position x_0 and initial velocity v_0.

(1) Find the position function $x(t)$ and determine whether the motion is *overdamped*, *critically damped*, or *underdamped*. If it is underdamped, write the position function in the form $x(t) = C_1 e^{-pt} \cos(\omega_1 t - \alpha_1)$.

(2) Find the undamped position function $u(t) = C_0 \cos(\omega_0 t - \alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c = 0$).

(3) Construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$.

(a) $m = 1, c = 4, k = 3; x_0 = 2, v_0 = 0$

ANS: (1) With damping.

$$x'' + 4x' + 3x = 0, \quad x(0) = 2, \quad x'(0) = 0.$$

$$\Rightarrow r^2 + 4r + 3 = (r+1)(r+3) = 0$$

$$\Rightarrow r_1 = -1, \quad r_2 = -3.$$

Thus $x(t) = c_1 e^{-t} + c_2 e^{-3t}$

As $x(0) = 2$, $c_1 + c_2 = 2$

As $x'(0) = 0$, $x'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$

$$x'(0) = -c_1 - 3c_2 = 0$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 2 \\ -c_1 - 3c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -1 \end{cases}$$

Since we have two real solutions $r_1 \neq r_2$

Thus $x(t) = 3e^{-t} - e^{-3t}$, which describes overdamped motion

(2) Without damping $(c=0)$

We have $x'' + 3x = 0$.

The char. eqn is $r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}\, i$.

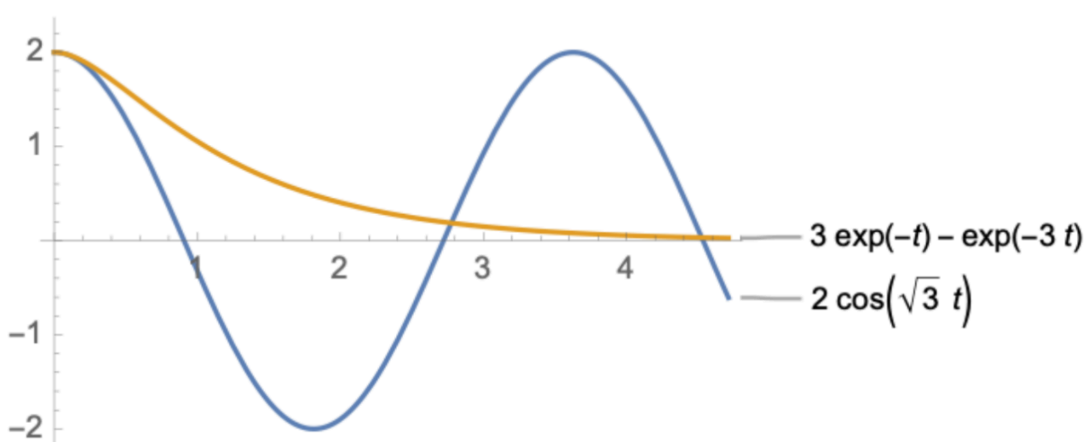
Thus $x(t) = A\cos\sqrt{3}t + B\sin\sqrt{3}t$.

$x'(t) = -\sqrt{3}A\sin\sqrt{3}t + \sqrt{3}B\cos\sqrt{3}t$.

$x(0) = 2, \quad A = 2$

$x'(0) = 0, \quad \sqrt{3}B = 0, \quad \Rightarrow B = 0$

Thus $x(t) = 2\cos\sqrt{3}t$ $\rightarrow$ Note this solution is already in the form $C_0\cos(\omega_0 t - \alpha_0)$



(b) $m = 1, c = 2, k = 10; x_0 = 2, v_0 = 4$

ANS: (1) With damping $c = 2$.

We have

$$x'' + 2x' + 10x = 0, \quad x(0) = 2, \quad x'(0) = 4.$$

The char. eqn is

$$r^2 + 2r + 10 = 0$$

$$\Rightarrow \quad r_{1,2} = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3i.$$

Thus $x(t) = e^{-t}(A\cos 3t + B\sin 3t)$

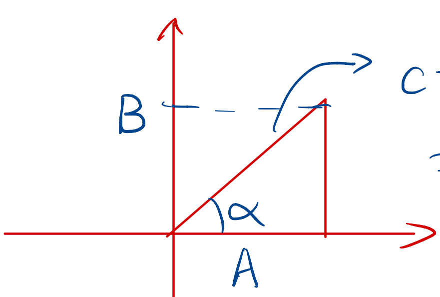
As $x(0) = 2$, $A = 2$.

$$x'(t) = -e^{-t}(A\cos 3t + B\sin 3t) + e^{-t}(-3A\sin 3t + 3B\cos 3t)$$

As $x'(0) = 4$, $-A + 3B = 4 \Rightarrow B = 2$.

So $x(t) = e^{-t}(\overset{A}{2}\cos 3t + \overset{B}{2}\sin 3t)$ underdamped

since the char. eqn has complex conjugates solutions.



$$c = \sqrt{A^2 + B^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$x(t) = 2\sqrt{2}\, e^{-t}\cos\left(3t - \frac{\pi}{4}\right)$$

$$\tan\alpha = \frac{2}{2} = 1$$

Notice α is in the 1st quadrant, $\alpha = \frac{\pi}{4}$

(2) Without damping $(c=0)$.

We have

$$x'' + 10x = 0, \quad x(0) = 2, \quad x'(0) = 4.$$

$$\Rightarrow r^2 + 10 = 0 \Rightarrow r = \pm\sqrt{10}\, i.$$

Then $x(t) = A\cos\sqrt{10}\,t + B\sin\sqrt{10}\,t$.

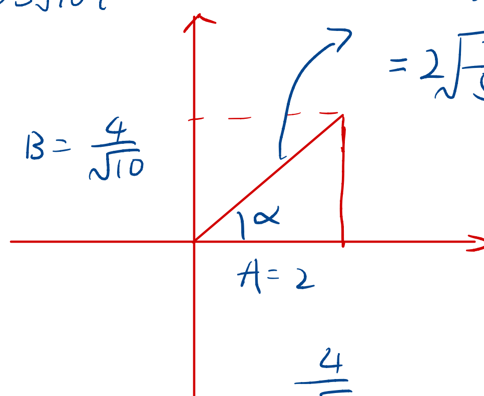
As $x(0) = 2$, $A = 2$

As $x'(0) = 4$, $x'(t) = -\sqrt{10}\,A\sin\sqrt{10}\,t + \sqrt{10}\,B\cos\sqrt{10}\,t$

Thus $\sqrt{10}\,B = 4 \Rightarrow B = \frac{4}{\sqrt{10}}$

So $x(t) = \underset{A}{2}\cos\sqrt{10}\,t + \underset{B}{\frac{4}{\sqrt{10}}}\sin\sqrt{10}\,t$

$$C = \sqrt{2^2 + \left(\frac{4}{\sqrt{10}}\right)^2} = 2\sqrt{\frac{7}{5}}$$



Thus $x(t) = 2\sqrt{\frac{7}{5}}\cos(\sqrt{10}\,t - 0.569)$

$$\tan\alpha = \frac{\frac{4}{\sqrt{10}}}{2} = \frac{2}{\sqrt{10}}$$

$\alpha \approx 0.5639$