3.3 Homogeneous Equations with Constant Coefficients

Review: Recall in Section 3.1, we talked about
$\text{2nd-order homogeneous equations with constant coefficients}$ of the following form

$$ay'' + by' + cy = 0$$

To solve for y, we first solve for r from the **characteristic equation**

$$ar^2 + br + c = 0,$$

which has roots $r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Case 1. r_1, r_2 are real and $r_1 \neq r_2$ ($b^2 - 4ac > 0$):

General solution: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2. r_1, r_2 are real and $r_1 = r_2$ ($b^2 - 4ac = 0$):

General solution: $y = (c_1 + c_2 x) e^{r_1 x}$

Case 3. r_1, r_2 are complex numbers ($b^2 - 4ac < 0$): (Not covered in Section 3.1 and 3.2)

We can write $r_{1,2} = A \pm Bi$.

General solution: $y = e^{Ax}(c_1 \cos Bx + c_2 \sin Bx)$

In this lecture, we will discuss how to solve the general $\text{homogeneous equations with constant coefficients}$ of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \tag{1}$$

Similar to 2nd-order homogeneous equations, we look at the corresponding **characteristic equation:**

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0 \tag{2}$$

We have 3 cases of the roots for Eq (2).

1. Distinct real roots
2. Repected real roots
3. Complex roots
    - distinct
    - repeated

Case 1. Distinct Real Roots

If the roots $r_1, r_2, \cdots, r_n$ of $Eq(2)$ are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

Example 1 Find the general solution to the given differential equation.

$$y^{(3)} - 7y'' + 12y' = 0$$

ANS: The corresponding char. eqn is

$$r^3 - 7r^2 + 12r = 0$$

$$\Rightarrow r(r^2 - 7r + 12) = 0$$

$$\Rightarrow r(r-3)(r-4) = 0$$

$$\Rightarrow r_1 = 0, \quad r_2 = 3, \quad r_3 = 4$$

So the general solution is

$$y(x) = C_1 \overbrace{\cancel{e^{0 \cdot x}}}^{1} + C_2 e^{3 \cdot x} + C_3 e^{4x}$$

$$\Rightarrow \boxed{y(x) = C_1 + C_2 e^{3x} + C_3 e^{4x}}$$

Case 2. Repeated Real Roots

If Eq (2) has repeated root r with multiplicty k, then the part of a general solution of $Eq(1)$ corresponds to r is

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{rx}$$

Example 2 Find a general solution the differential equation.

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = 0$$

ANS: The corresponding char. eqn is

$$r^4 + 3r^3 + 3r^2 + r = 0$$

$$\Rightarrow r(r^3 + 3r^2 + 3r + 1) = 0 \quad ⊗$$

Notice that $r=-1$ is a solution since

$$(-1)^3 + 3 + 3(-1) + 1 = 0$$

$$⊗ \Rightarrow r(r-(-1))(r^2 + 2r + 1) = 0$$

Long division of polynomials

$$\begin{array}{r|l} & r^2 + 2r + 1 \\ \hline r+1 & r^3 + 3r^2 + 3r + 1 \\ & r^3 + r^2 \\ \hline & 2r^2 + 3r + 1 \\ & 2r^2 + 2r \\ \hline & r + 1 \\ & r + 1 \\ \hline & 0 \end{array}$$

$$\Rightarrow r(r+1)(r+1)^2 = 0$$

$$\Rightarrow r(r+1)^3 = 0$$

$$\Rightarrow r_1 = 0, \quad r_2 = r_3 = r_4 = -1 \quad \text{(Repeated real roots)}$$

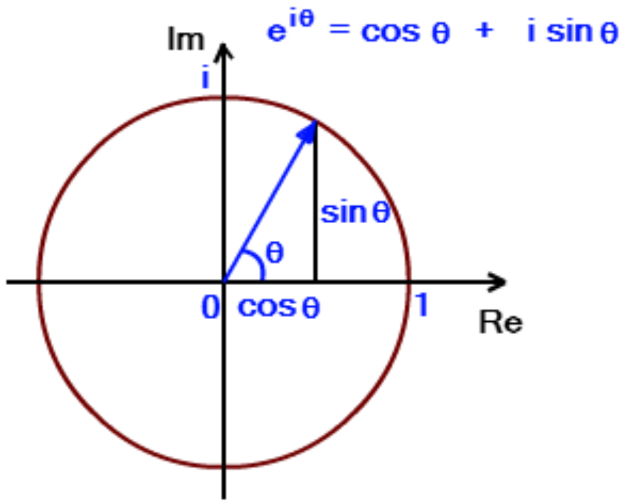
Thus the general solution is

$$y(x) = C_1 \cancel{e^{0\cdot x}}^{\,1} + (C_2 + C_3 x + C_4 x^2) e^{-1\cdot x}$$

$$\Rightarrow \boxed{y(x) = C_1 + (C_2 + C_3 x + C_4 x^2) e^{-x}}$$

Euler's Formula for Complex Numbers $i = \sqrt{-1}$

- **Euler's formula**: $e^{i\theta} = \cos\theta + i\sin\theta$



- $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$, where $z = x + iy$ is any complex number.

> **Case 3. Complex Roots**
>
> **Unrepeated complex roots:** If $r_{1,2} = A \pm Bi$ are roots of the characteristic equation, then the corresponding part to the general solution
>
> $$y = e^{Ax}(c_1 \cos Bx + c_2 \sin Bx)$$

Remark: We have the above formula since

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$= C_1 e^{(A+Bi)x} + C_2 e^{(A-Bi)x} = C_1 e^{Ax} e^{Bix} + C_2 e^{Ax} e^{-Bix}$$

$$= C_1 e^{Ax} \cdot (\cos Bx + i \sin Bx) + C_2 e^{Ax} (\cos Bx - i \sin Bx)$$
$$= e^{Ax}\left[(C_1 + C_2)\cos Bx + i\,(C_1 - C_2)\sin Bx\right]$$

$$= e^{Ax}(c_1 \cos Bx + c_2 \sin Bx)$$

Example 3 Find the general solutions of the differential equation.

$$3y^{(4)} + 7y'' + 4y = 0$$

ANS: The corresponding char. eqn is

$$3r^4 + 7r^2 + 4 = 0$$

$$\Rightarrow 3x^2 + 7x + 4 = 0 \quad (\text{Let } x = r^2)$$

$$\Rightarrow x = \frac{-7 \pm \sqrt{7^2 - 4\times3\times4}}{6} = \frac{-7 \pm 1}{6}$$

$$\Rightarrow x = r^2 = -1 \quad \text{or} \quad x = r^2 = -\frac{4}{3}$$

So $r_{1,2} = \pm\sqrt{-1} = \pm i \Rightarrow \begin{cases} A = 0 \\ B = 1 \end{cases}$

$(= A \pm Bi)$

$r_{3,4} = \pm\sqrt{-\frac{4}{3}} = \pm\sqrt{\frac{4}{3}\times(-1)} = \pm\frac{2}{\sqrt{3}} i \Rightarrow \begin{cases} A = 0 \\ B = \frac{2}{\sqrt{3}} \end{cases}$

(complex roots)

So $y(x) = \cancel{e^{0\cdot x}}^{\,1}(C_1 \cos x + C_2 \sin x) + \cancel{e^{0\cdot x}}^{\,1}\left(C_3 \cos\frac{2}{\sqrt{3}}x + C_4 \sin\frac{2}{\sqrt{3}}x\right)$

$$\Rightarrow y(x) = C_1 \cos x + C_2 \sin x + C_3 \cos\frac{2}{\sqrt{3}}x + C_4 \sin\frac{2}{\sqrt{3}}x$$

Repeated complex roots

If the conjugate pair $a \pm bi$ has multiplicity k, then the corresponding part of the general solution has the form

$$\begin{aligned}&\left(A_1 + A_2 x + \cdots + A_k x^{k-1}\right) e^{(a+bi)x} + \left(B_1 + B_2 x + \cdots + B_k x^{k-1}\right) e^{(a-bi)x} \\ &= \sum_{p=0}^{k-1} x^p e^{ax} \left(c_p \cos bx + d_p \sin bx\right)\end{aligned}$$

Example 4 In the following question, one solution of the differential equation is given. Find the general solution.

$$3y^{(3)} - 2y'' + 12y' - 8y = 0; y = e^{2x/3} = e^{rx}$$

ANS: The char. eqn is

$$3r^3 - 2r^2 + 12r - 8 = 0 \cdots \circledast$$

Since $y(x) = e^{\frac{2}{3}x}$ is a solution, of eqn,

$r = \frac{2}{3}$ is a root of $\circledast$

So $(r - \frac{2}{3})$ is a factor of the LHS of $\circledast$

$$\begin{array}{r|l} & 3r^2 + 12 \\ \hline r - \frac{2}{3} & 3r^3 - 2r^2 + 12r - 8 \\ & 3r^3 - 2r^2 \\ \hline & 12r - 8 \\ & 12r - 8 \\ \hline & 0 \end{array}$$

$$\circledast \Rightarrow (r - \tfrac{2}{3})(3r^2 + 12) = 0$$

$$\Rightarrow r_1 = \frac{2}{3}, \quad r_{2,3} = \pm\sqrt{-4} = \pm 2i \Rightarrow \begin{cases} A = 0 \\ B = 2 \end{cases}$$

$$y(x) = c_1 e^{\frac{2}{3}x} + \cancel{e^{0 \cdot x}}^{\,1} (C_2 \cos 2x + C_3 \sin 2x)$$

$$\Rightarrow \boxed{y(x) = C_1 e^{\frac{2}{3}x} + C_2 \cos 2x + C_3 \sin 2x}$$

Exercise 5 Find general solutions of the equations in the following question. First find a small integral root of the characteristic equation by inspection; then factor by division.

$$y^{(3)} + y'' - 2y = 0$$

Long division of poly

$$\begin{array}{r} r^2 + 2r + 2 \\ r-1 \,\big)\, r^3 + r^2 - 2 \\ \underline{r^3 - r^2} \\ 2r^2 - 2 \\ \underline{2r^2 - 2r} \\ 2r - 2 \\ \underline{2r - 2} \\ 0 \end{array}$$

ANS: The corresponding char. eqn. is

$$r^3 + r^2 - 2 = 0 \quad \circledast$$

Notice that $r = 1$ is a solution.

We can factor $r-1$ from $r^3 + r^2 - 2$

So $\circledast \Rightarrow (r-1)(r^2+2r+2) = 0$

Thus $r_1 = 1$, $r_{2,3} = \dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm \sqrt{-4}}{2} = \dfrac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm i \Rightarrow \begin{cases} A = -1 \\ B = 1 \end{cases}$

Thus the general solution is

$$y(x) = C_1 e^{x} + e^{-x}\left(C_2 \cos x + C_3 \sin x\right)$$

Euler equations

According to our handwritten HW#5 Problem 51 in Section 3.1, the substitution $v = \ln x (x > 0)$ transforms the second-order Euler equation $ax^2y'' + bxy' + cy = 0$ to a constant-coefficient homogeneous linear equation. Similarly, the same substitution transforms the third-order Euler equation

$$ax^3y''' + bx^2y'' + cxy' + dy = 0$$

(where a, b, c, d are constants) into the constant-coefficient equation

$$a\frac{d^3y}{dv^3} + (b-3a)\frac{d^2y}{dv^2} + (c-b+2a)\frac{dy}{dv} + dy = 0$$

Example 6 Use substitution $v = \ln x$ from above to find general solutions (for $x > 0$) of the following Euler equation.

$a=1,\quad b=-3,\quad c=1,\quad d=0$

$$x^3y''' - 3x^2y'' + xy' = 0 \quad \otimes$$

ANS: Let $v = \ln x$, by the above discussion, we can write $\otimes$ as

$$a\frac{d^3y}{dv^3} + (b-3a)\frac{d^2y}{dv^2} + (c-b+2a)\frac{dy}{dv} + dy = 0$$

$$\Rightarrow \frac{d^3y}{dv^3} + (-3-3)\frac{d^2y}{dv^2} + (1+3+2)\frac{dy}{dv} = 0$$

$$\Rightarrow \frac{d^3y}{dv^3} - 6\frac{d^2y}{dv^2} + 6\frac{dy}{dv} = 0$$

The char. eqn. of the above eqn is

$$r^3 - 6r^2 + 6r = 0$$

$$\Rightarrow r(r^2 - 6r + 6) = 0$$

$$\Rightarrow r_1 = 0, \quad r_{2,3} = \frac{6 \pm \sqrt{6^2 - 6\times 4}}{2} = \frac{6 \pm \sqrt{12}}{2} = 3 \pm \sqrt{3}$$

(distinct real)

$$y(v) = c_1 \cancel{e^{0 \cdot v}} + c_2 e^{(3+\sqrt{3})v} + c_3 e^{(3-\sqrt{3})v}$$

$$\Rightarrow y(x) = C_1 + C_2 e^{(\ln x)(3+\sqrt{3})} + C_3 e^{(\ln x)(3-\sqrt{3})}$$

$$\Rightarrow y(x) = C_1 + C_2 x^{3+\sqrt{3}} + C_3 x^{3-\sqrt{3}}$$