# 3.2 General Solutions of Linear Equations


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### 1. Linearly Independent Solutions

#### 1.1. Definition of linearly dependent/independent

The $n$ functions $f_1, f_2, \cdots, f_n$ are said to be linearly dependent on the interval $I$ if there exist constants $c_1, c_2, \cdots, c_n$ not all zero such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

for all $x$ in $I$.

The $n$ functions $f_1, f_2, \cdots, f_n$ are said to be linearly independent on the interval $I$ if they are not linearly dependent. Equivalently, they are linearly independent on $I$ if

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

holds on $I$ only when

$$c_1 = c_2 = \cdots = c_n = 0.$$

**Example 1** Show **directly** that the given functions are linearly dependent on the real line.

(1) $f(x) = 3, \quad g(x) = 2\cos^2 x, \quad h(x) = \cos 2x$

(2) $f(x) = 5, \quad g(x) = 2 - 3x^2, \quad h(x) = 10 + 15x^2$ (exercise)

ANS: We need to find $C_1$, $C_2$, $C_3$ not all zeros, such that

$$C_1 f(x) + C_2 g(x) + C_3 h(x) = 0$$

$2\cos^2 x = \cos 2x + 1$

$$\Rightarrow C_1 \cdot 3 + C_2 \cdot \underbrace{(2\cos^2 x)}_{= \cos 2x + 1} + C_3 \cdot \cos 2x = 0$$

$$\Rightarrow 3C_1 + C_2 \cdot \cos 2x + C_2 + C_3 \cdot \cos 2x = 0$$

$$\Rightarrow (3C_1 + C_2) + (C_2 + C_3) \cdot \cos 2x = 0$$

We need $\begin{cases} 3C_1 + C_2 = 0 \\ C_2 + C_3 = 0 \end{cases}$

Let $C_2 = 1$, then $C_3 = -1$, $C_1 = -\frac{1}{3}$

Thus

$$-\frac{1}{3} \cdot \underbrace{3}_{f(x)} + 1 \cdot \underbrace{2\cos^2 x}_{g(x)} - 1 \cdot \underbrace{\cos 2x}_{h(x)} = 0$$

i.e. $-\frac{1}{3} \cdot f(x) + 1 \cdot g(x) - 1 \cdot h(x) = 0$

Thus $f(x)$, $g(x)$, $h(x)$ are linearly dependent.

### 1.2. Wronskian of $n$ functions

Suppose that the $n$ functions $f_1, f_2, \cdots, f_n$ are all $n-1$ times differentiable. Then their **Wronskian** is the $n \times n$ determinant

$$W(x) = W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

- The Wronskian of $n$ **linearly dependent** functions $f_1, f_2, \cdots, f_n$ is **identically zero**.

  Idea of the proof:

  - We show for the case $n = 2$. The case for general $n$ is similar.
  - If $f_1$ and $f_2$ are linearly dependent, then $c_1 f_1 + c_2 f_2 = 0 \quad (*)$ has nontrivial solutions for $c_1$ and $c_2$ ($c_1$ and $c_2$ are not all zeros).
  - We also have $c_1 f_1' + c_2 f_2' = 0$ from $(*)$.
  - Thus we have the linear system of equations

    $$\begin{aligned} c_1 f_1 + c_2 f_2 &= 0 \\ c_1 f_1' + c_2 f_2' &= 0 \end{aligned}$$

  - By a theorem in linear algebra, the above system of equations has nontrivial solutions if and only if

    $$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$$

- So to show that the functions $f_1, f_2, \cdots, f_n$ are **linearly independent** on the interval $I$, it suffices to show that their Wronskian is **nonzero at just one point of $I$**.

**Example 2** Use the Wronskian to prove that the given functions are linearly independent on the indicated interval.

$$f(x)=e^{x}, \quad g(x)=\cos x, \quad h(x)=\sin x; \quad \text{the real line}$$

**Remark:** $3 \times 3$ matrix determinant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a\,(ei-fh)-b(di-fg)+c(dh-eg)$$

ANS: By the previous page, we know it suffices to show that $W(f,g,h) \neq 0$ at just one point on the real line.

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix}$$

$$= e^x \cdot \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - \cos x \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} + \sin x \begin{vmatrix} e^x & -\sin x \\ e^x & -\cos x \end{vmatrix}$$

$$= e^x\,(\underbrace{\sin^2 x + \cos^2 x}_{1}) - \cos x\,(-e^x \sin x - e^x \cos x) + \sin x\,(-e^x \cos x + e^x \sin x)$$

$$= e^x + \cancel{e^x \cos x \sin x} + \underline{e^x \cos^2 x} - \cancel{e^x \sin x \cos x} + \underline{e^x \sin^2 x}$$

$$= 2e^x \quad \text{is never zero on the real line.}$$

$\neq 0$

So $f(x)$, $g(x)$, $h(x)$ are linearly independent.

**2. *n*th-order linear differential equation**

The general ***n*th-order linear** differential equation is of the form

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x).$$

We assume that the coefficient functions $P_i(x)$ and $F(x)$ are continuous on some open interval $I$.

### 2.1 homogeneous linear equation

Similar to Section 3.1, we consider the **homogeneous linear equation**

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \qquad (1)$$

**THEOREM 1 Principle of Superposition for Homogeneous Equations**

Let $y_1, y_2, \cdots y_n$ be $n$ solutions of the homogeneous linear equation (1) on the interval I. If $c_1, c_2, \cdots, c_n$ are constants, then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is also a solution of Eq. (1) on $I$.

**THEOREM 4 General Solutions of Homogeneous Equations**

Let $y_1, y_2, \cdots y_n$ be $n$ linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \qquad (1)$$

on an open interval I where the $p_i$ are continuous. If $Y$ is any solution of Eq. (1), then there exist numbers $c_1, c_2, \cdots, c_n$ such that

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

for all $x$ in $I$.

**Example 3** In the following question, a third-order homogeneous linear equation and three linearly independent solutions are given. Find a particular solution satisfying the given initial conditions.

$$x^3y^{(3)} + 6x^2y'' + 4xy' - 4y = 0;$$

$$y(1) = 1, y'(1) = 5, \quad y''(1) = -11,$$

$$y_1 = x, \quad y_2 = x^{-2}, y_3 = x^{-2}\ln x$$

ANS: By Thm 4, we know

$$y(x) = c_1y_1 + c_2y_2 + c_3y_3$$

is a general solution. i.e.

$$y(x) = c_1x + c_2x^{-2} + c_3x^{-2}\ln x$$

Since $y(1) = 1$,

$$y(1) = c_1 + c_2 + c_3 \cdot \cancel{\ln 1}^{0} = c_1 + c_2 = 1$$

Since $y'(1) = 5$,

$$y'(x) = c_1 - 2c_2x^{-3} + c_3(-2x^{-3}\ln x + x^{-3})$$

$$y'(1) = c_1 - 2c_2 + c_3 \cdot 1 = 5$$

$$= c_1 - 2c_2 + c_3 = 5$$

Since $y''(1) = -11$

$$y''(x) = 6c_2x^{-4} + c_3(6x^{-4}\ln x - 2x^{-4} - 3x^{-4})$$

$$= 6c_2x^{-4} + c_3(6x^{-4}\ln x - 5x^{-4})$$

$$y''(1) = 6c_2 - 5c_3 = -11$$

So $\begin{cases} c_1 + c_2 = 1 \Rightarrow c_1 = 1 - c_2 \\ c_1 - 2c_2 + c_3 = 5 \\ 6c_2 - 5c_3 = -11 \end{cases} \Rightarrow \begin{cases} (-3c_2 + c_3 = 4) \times 2 \\ 6c_2 - 5c_3 = -11 \end{cases} \Rightarrow -3c_3 = -3$

$\Rightarrow C_3 = 1$

$-3C_2 + 1 = 4 \Rightarrow C_2 = -1$

$C_1 = 1 - C_2 = 1 - (-1) = 2$.

Thus $y = 2x - x^{-2} + x^{-2}\ln x$ is a particular solution of the given initial value problem.

## The method of reduction of order

Suppose that one solution $y_1(x)$ of the homogeneous second-order linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \qquad (3)$$

is known (on an interval $I$ where $p$ and $q$ are continuous functions). The method of **reduction of order** consists of substituting $y_2(x) = v(x)y_1(x)$ in (3) and determine the function $v(x)$ so that $y_2(x)$ is a second linearly independent solution of (2).

After substituting $y_2(x) = v(x)y_1(x)$ in Eq. (3), use the fact that $y_1(x)$ is a solution. We can deduce that

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

We can solve this for $v$ to find the solution $y_2(x)$ of equation (3).

**Example 4** Consider the equation

$$x^2y'' - 5xy' + 9y = 0 \ (x > 0),$$

Notice that $y_1(x) = x^3$ is a solution. Subsitute $y = vx^3$ and deduce that $xv'' + v' = 0$. Solve this equation and obtain the second solution $y_2(x) = x^3 \ln x$.

ANS: We write the given equation in the form of Eq (3).

$$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0 \quad \circledast$$

Let $y_2 = v y_1 = v x^3$

$$y_2' = \underline{v'x^3} + \underline{3vx^2}$$

$$y_2'' = \underline{v''x^3 + 3v'x^2} + \underline{3v'x^2 + 6vx}$$

$$= v''x^3 + 6v'x^2 + 6xv$$

Plug them into $\circledast$

$$(v''x^3 + 6v'x^2 + 6xv) - \frac{5}{x}(v'x^3 + 3vx^2) + \frac{9}{x^2}v \cdot x^3 = 0$$

$$\Rightarrow v''x^3 + \underline{6v'x^2} + 6xv - \underline{5x^2v'} - 15vx + 9vx = 0$$

$$\Rightarrow v''x^3 + v'x^2 = 0$$

$$\Rightarrow v''x + v' = 0$$

Let $u = v'$, then $v'' = u'$. thus

$$u'x + u = 0 \Rightarrow \frac{du}{dx} x + u = 0$$

$$\Rightarrow \frac{du}{dx} \cdot x = -u \Rightarrow \int \frac{du}{u} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln|u| = -\ln|x| + C$$

$$\Rightarrow u = c_1 e^{-\ln x} = \frac{c_1}{x}$$

$$\frac{dv}{dx} = v' = u = \frac{c_1}{x}$$

$$\Rightarrow \frac{dv}{dx} = \frac{c_1}{x} \Rightarrow v(x) = c_1 \ln x + c_2$$

Let $c_1 = 1$, $c_2 = 0$, then $v(x) = \ln x$

Thus $y_2(x) = v(x) y_1(x) = x^3 \ln x$

### 2.2. Nonhomogeneous Equations

Now we consider the *nonhomogeneous* *n*th-order linear differential equation

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_{n-1}(x)y'+p_n(x)y=f(x) \tag{4}$$

with associated homogeneous equation

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_{n-1}(x)y'+p_n(x)y=0 \tag{5}$$

**THEOREM 5 Solutions of Nonhomogeneous Equations**

Let $y_p$ be a particular solution of the nonhomogeneous equation in (4) on an open interval $I$ where the functions $p_i$ and $f$ are continuous. Let $y_1, y_2, \cdots y_n$ be linearly independent solutions of the associated homogeneous equation in (5). If Y is any solution whatsoever of Eq. (4) on I, then there exist numbers $c_1, c_2, \cdots, c_n$ such that

$$Y(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n+y_p(x)=y_c+y_p$$

for all $x$ in $I$.

**Exercise 5** Notice that $y_p=3x$ is a particular solution of the equation

$$y''+4y=12x$$

and that $y_c(x)=c_1\cos 2x+c_2\sin 2x$ is its complementary solution. Find a solution of the given equation that satisfies the initial conditions $y(0)=5, y'(0)=7$.

ANS: By Thm5, we have

$$y(x)=y_c+y_p=c_1\cos 2x+c_2\sin 2x+3x$$

is a general solution

Since $y(0)=5$,

$$y(0)=c_1=5$$

Since $y'(x)=-2c_1\sin 2x+2c_2\cos 2x+3$

$y'(0)=2c_2+3=7 \Rightarrow c_2=2$

Thus

$$y(x)=5\cos 2x+2\sin 2x+3x$$