3.1 Introduction: Second-Order Linear Equations

1. Definition of second-order linear equations

A *linear second-order equation* can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) \tag{1}$$

We assume that $A(x), B(x), C(x)$ and $F(x)$ are continuous functions on some open interval I.

For example,

$$e^x y'' + (\cos x)y' + (1 + \sqrt{x})y = \tan^{-1} x$$

is linear because the dependent variable y and its derivatives y' and y'' appear linearly.

The equations

$$y'' = yy' \quad \text{and} \quad y'' + 2(y')^2 + 4y^3 = 0$$

are **not** linear because products and powers of y or its derivatives appear.

2. Homogeneous Second-Order Linear Equations

If the function $F(x) = 0$ on the right-hand side of Eq. (1), then we call Eq. (1) a **homogeneous** linear equation; otherwise, it is **nonhomogeneous**. In general, the homogeneous linear equation associated with Eq. (1) is

$$A(x)y'' + B(x)y' + C(x)y = 0 \tag{2}$$

For example, the second-order equation

$$2x^2y'' + 2xy' + 3y = \sin x$$

is nonhomogeneous; its associated homogeneous equation is

$$2x^2y'' + 2xy' + 3y = 0$$

Consider

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

Assume that $A(x) \neq 0$ at each point of the open interval I, we can divide each term in Eq. (1) by $A(x)$ and write it in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

We will discuss first the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

Theorem 1 Principle of Superposition for Homogeneous Equations

Let y_1 and y_2 be two solutions of the homogeneous linear equation in Eq. (3) on the interval I. If c_1 and c_2 are constants, then the linear combination

$$y = c_1y_1 + c_2y_2$$

is also a solution of Eq. (3) on I.

$$c_1y_1'' + c_1p(x)y_1' + c_1q(x)y_1 = 0$$
$$+ \quad c_2y_2'' + c_2p(x)y_2' + c_2q(x)y_2 = 0$$
$$\underline{c_1y_1'' + c_2y_2''} + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2) = 0$$
$$\Rightarrow y'' + p(x)y' + q(x)y = 0$$

Application of Theorem 1. In Examples 1 and Exercise 2, a homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions.

Example 1

$$y'' - 3y' + 2y = 0; \quad y_1 = e^x, \quad y_2 = e^{2x}; \quad y(0) = 1, \quad y'(0) = 7.$$

ANS: If $y_1 = e^x$, then $y_1' = e^x$, $y_1'' = e^x$

$$y_1'' - 3y_1' + 2y_1 = e^x - 3e^x + 2e^x = 0$$

So y_1 is a solution

If $y_2 = e^{2x}$, then $y_2' = 2e^{2x}$, $y_2'' = 4e^{2x}$

$$y_2'' - 3y_2' + 2y_2 = 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$$

So y_2 is also a solution.

By Theorem 1, we know

$y = c_1y_1 + c_2y_2 = c_1e^x + c_2e^{2x}$ is also a solution of $\otimes$

Since $y(0) = 1$, $y'(0) = 7$

$$\left.\begin{aligned} y(0) &= c_1e^0 + c_2e^{2\cdot 0} = \underline{c_1 + c_2 = 1} \\ y'(x) &= c_1e^x + 2c_2e^{2x} \\ y'(0) &= c_1e^0 + 2c_2e^{2\cdot 0} = \underline{c_1 + 2c_2 = 7} \end{aligned}\right\} \Rightarrow \begin{cases} c_1 = -5 \\ c_2 = 6 \end{cases}$$

Thus $\boxed{y(x) = -5e^{x} + 6e^{2x}}$

Exercise 2

$$x^2y'' - 2xy' + 2y = 0; \quad \circledast \quad y_1 = x, \quad , y_2 = x^2; \quad y(1) = 3, \quad y'(1) = 1.$$

ANS: . If $y_1 = x$, then $y_1' = 1$, $y_1'' = 0$

Then $x^2 \cdot y_1'' - 2x y_1' + 2y_1 = x^2 \cdot 0 - 2x \cdot 1 + 2x = 0$

Thus y_1 is a solution for $\circledast$.

If $y_2 = x^2$, then $y_2' = 2x$, $y_2'' = 2$.

Then $x^2 \cdot y_2'' - 2x \cdot y_2' + 2 \cdot y_2 = x^2 \cdot 2 - 2x \cdot 2x + 2x^2 = 0$

Thus y_2 is a solution for $\circledast$.

By thm 1. $y = C_1 y_1 + C_2 y_2$ is a solution for $\circledast$

$= C_1 x + C_2 \cdot x^2$.

Since $y(1) = 3$, $y(1) = C_1 + C_2 = 3$

Since $y'(1)$; $y'(x) = C_1 + 2C_2 x$, $y'(1) = C_1 + 2C_2 = 1$

$$\begin{cases} C_1 + C_2 = 3 \\ C_1 + 2C_2 = 1 \end{cases} \Rightarrow \begin{cases} C_1 = 5 \\ C_2 = -2 \end{cases}$$

Thus $y(x) = 5x - 2x^2$ is a particular solution for the given initial value problem.

Theorem 2 Existence and Uniqueness for Linear Equations

Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

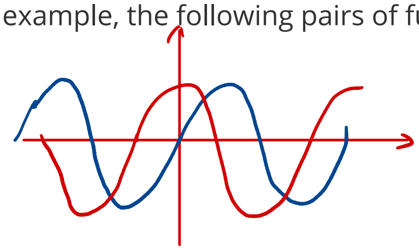
has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

3. Linear Independence of Two Functions

Two functions defined on an open interval I are said to be **linearly independent** on I if neither is a constant multiple of the other. Two functions are said to be **linearly dependent** on an open interval if one of them is a constant multiple of the other. $y_1 = x$, $y_2 = 5x$, y_1 and y_2 are linearly dependent since $y_2 = 5y_1$

For example, the following pairs of functions are linearly independent on the entire real line



$\sin x$ and $\cos x$

e^x and xe^x

$x + 1$ and x^3

The functions $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$ are linearly dependent.

$f(x) = 2\sin x \cos x = 2g(x)$

We can compute the **Wronskian** of two functions to determine if they are linearly independent (or dependent).

Given two functions f and g, the **Wronskian** of f and g is the determinant

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

For example,

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

and

$$W(x, 5x) = \begin{vmatrix} x & 5x \\ 1 & 5 \end{vmatrix} = 5x - 5x = 0.$$

Theorem 3 Wronskians of Solutions

Suppose that $y_1$and y_2 are two solutions of the homogeneous second-order linear equation Eq. (3)

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I on which p and q are continuous.
(a) If $y_1$and $y_2$are linearly dependent, then $W(y_1, y_2) \equiv 0$ on I.
(b) If $y_1$and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of I.

Theorem 4 General Solutions of Homogeneous Equations

Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation Eq. (3)

$$y'' + p(x)y' + q(x)y = 0$$

with p and q continuous on the open interval I. If Y is any solution whatsoever of Eq. (3) on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all x in I.

4. Linear Second-Order Equations with Constant Coefficients

Let's discuss how to solve the homogeneous second-order linear differential equation

$$ay'' + by' + cy = 0 \tag{4}$$

with constant coefficients a, b, and c.

Consider a function of the form $y = e^{rx}$. Observe that

$$y' = (e^{rx})' = re^{rx}, \qquad \text{and} \quad y'' = (e^{rx})'' = r^2 e^{rx}.$$

This suggest that we can try to find r such that when we substitute y, y' and y'' into Eq. (4), we will get zero on the left hand-side. $ar^2e^{rx} + bre^{rx} + ce^{rx} = e^{rx}(ar^2 + br + c) = 0 \Rightarrow ar^2 + br + c = 0$

Example 3 Find the values of r such that $y(x) = e^{rx}$ is a solution of the given differential equation.

$$y'' + 2y' - 15y = 0$$

ANS: If $y(x) = e^{rx}$, then $y' = re^{rx}$, $y'' = r^2e^{rx}$

So we need to find r such that

$$r^2e^{rx} + 2re^{rx} - 15e^{rx} = 0$$

$$\Rightarrow e^{rx}(r^2 + 2r - 15) = 0$$

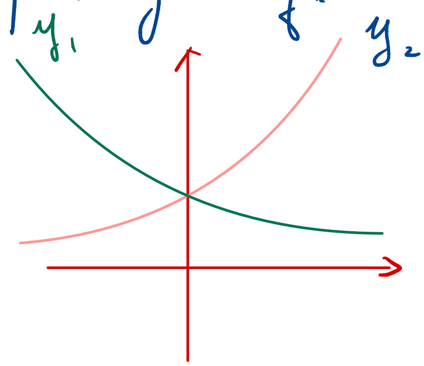
Note $e^{rx} \neq 0$ for any x.

So we have

$$r^2 + 2r - 15 = 0 \quad \text{(characteristic eqn)}$$

$$\Rightarrow (r+5)(r-3) = 0 \Rightarrow r = -5 \text{ or } r = 3.$$

So $y_1 = e^{-5x}$ and $y_2 = e^{3x}$ are solutions of the given eqn.



Note y_1 and y_2 are linealy independent.

By Thm 4. $y(x) = c_1y_1 + c_2y_2 = c_1e^{-5x} + c_2e^{3x}$

is general solution, where c_1 and c_2 are constants.

In general, we subsititute $y = e^{rx}$ in Eq. (4). Then

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

Since e^{rx} is never zero. We conclude $y = e^{rx}$ will satisfy the differential equation in Eq. (4) precisely when r is a root of the algebraic equation

$$ar^2 + br + c = 0 \tag{5}$$

This quadratic equation is called the **characteristic equation** of the homogeneous linear differential equation

$$ay'' + by' + cy = 0$$

If Eq. (5) has distinct (unequal) roots r_1 and r_2, then the corresponding solutions $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ of Eq. (5). are linearly independent. Why?

By looking at their graph or computing $W(y_1, y_2) (\neq 0)$

Theorem 5 Distinct Real Roots

If the roots r_1 and r_2 of the characteristic equation in Eq. (5) are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution of Eq. (4).

Question: What if we have $r_1 = r_2$ for the characeristic equation?

Example 4

Find general solutions of the given differential equations.

$$y'' + 4y' + 4y = 0 \quad \circledast$$

ANS: The corresponding char. eqn is

$$r^2 + 4r + 4 = 0$$

$$\Rightarrow (r+2)^2 = 0 \Rightarrow r_1 = r_2 = -2$$

So $y_1 = e^{r_1 x} = e^{r_2 x} = e^{-2x}$ is a solution to $\circledast$.

How do we find another solution y_2 such that y_1 & y_2 are linearly independent?

Let's check if $y_2 = xe^{-2x}$ $(= xy_1)$ works.

$$y_2' = (xe^{-2x})' = x(e^{-2x})' + (x)'e^{-2x} = -2xe^{-2x} + e^{-2x}$$

$$y_2'' = -2e^{-2x} + 4xe^{-2x} - 2e^{-2x} = -4e^{-2x} + 4xe^{-2x}$$

$$y_2'' + 4y_2' + 4y_2 = -4e^{-2x} + 4xe^{-2x} + 4(-2xe^{-2x} + e^{-2x}) + 4xe^{-2x}$$
$$= 0$$

So $y_2 = xe^{-2x}$ is a solution. And $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$ are linearly independent.

By Thm 4, $y(x) = C_1 y_1 + C_2 y_2 \Rightarrow y(x) = (C_1 + C_2 x)e^{-2x}$ is a general solution.

In general, we have the following theorem if $r_1 = r_2$.

Theorem 6 Repeated Roots

If the characteristic equation in Eq. (5) has equal (necessarily real) roots $r_1 = r_2$, then,

$$y(x) = (c_1 + c_2 x)e^{r_1 x}$$

is a general solution of Eq. (5).

Example 5

Find general solutions of the given differential equations.

(1) $9y'' - 6y' + y = 0$

(2) $2y'' + 3y' = 0$ (exercise)

ANS: (1) The corresponding char. eqn is

$$9r^2 - 6r + 1 = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow r^2 - \frac{2}{3}r + \frac{1}{9} = 0$$

$$\Rightarrow \left(r - \frac{1}{3}\right)^2 = 0$$

$$\Rightarrow r_1 = r_2 = \frac{1}{3}$$

The general solution is $y = (c_1 + c_2 x)e^{\frac{1}{3}x}$, where

c_1 and c_2 are constants.

(2). The corresponding characteristic equation is

$$2r^2+3r=0$$

$$\Rightarrow r(2r+3)=0$$

$$\Rightarrow r=0 \text{ or } r=-\frac{3}{2} \text{ (distinct)}$$

So $y=c_1y_1+c_2y_2=c_1e^{0\cdot x}+c_2e^{-\frac{3}{2}x}=c_1+c_2e^{-\frac{3}{2}x}$

is a general solution.

Example 6. The equation

$$y(x) = c_1 + c_2 e^{-10x}$$

gives a general solution $y(x)$ of a homogeneous second-order differential equation $ay'' + by' + cy = 0$ with constant coefficients. Find such an equation.

ANS: $y(x) = C_1 + C_2 e^{-10x} = C_1 \cdot 1 + C_2 e^{-10x} = C_1 e^{\overset{r_1}{\overset{\shortparallel}{0}}x} + C_2 e^{\overset{r_2}{\overset{\shortparallel}{-10}}x}$

$$\Rightarrow \begin{cases} r_1 = 0 \\ r_2 = -10 \end{cases}$$

So $r_1 = 0$, $r_2 = -10$ are solutions to the char. eqn.

Thus $(r - \overset{r_1}{0})(r - (-\overset{r_2}{10})) = 0$

$\Leftrightarrow \quad r(r+10) = 0$

$\Leftrightarrow \quad r^2 + 10r = 0 \quad (\Leftrightarrow ar^2 + br + c = 0)$

is the char. eqn.

So $a = 1$, $b = 10$, $c = 0$.

Thus the diff eqn is

$$y'' + 10y' = 0$$

5. Euler Equation

A second-order Euler equation is one of the form

$$ax^2y'' + bxy' + cy = 0 \tag{8}$$

where a, b, c are constants.

Example 7. Make the substitution $v = \ln x$ of the following question to find general solutions (for $x > 0$) of the Euler equation.

$$x^2y'' + 2xy' - 12y = 0$$

ANS: Let $v = \ln x$.

$\frac{1}{x}$ since $v = \ln x$

$$y' = \frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dv}{dv} = \frac{dy}{dv} \cdot \frac{dv}{dx} = \frac{dy}{dv} \cdot \frac{1}{x}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dv}\right)$$

$$= -\frac{1}{x^2} \cdot \frac{dy}{dv} + \frac{1}{x} \cdot \frac{d}{dx}\left(\frac{dy}{dv}\right)$$

$$= -\frac{1}{x^2} \cdot \frac{dy}{dv} + \frac{1}{x} \cdot \frac{d}{dx} \cdot \frac{dv}{dv} \cdot \frac{dy}{dv}$$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x} \cdot \frac{d^2y}{dv^2} \cdot \frac{dv}{dx}$$ $\frac{1}{x}$ since $v = \ln x$

$$\Rightarrow \quad y'' = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2} \cdot \frac{d^2y}{dv^2}$$

Plug them into Eq ~~(9)~~ (8). we have.

$$x^2\left(-\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2} \cdot \frac{d^2y}{dv^2}\right) + 2x \cdot \frac{1}{x}\frac{dy}{dv} - 12y = 0$$

$$\Rightarrow -\frac{dy}{dv} + \frac{d^2y}{dv^2} + 2\frac{dy}{dv} - 12y = 0$$

$$\Rightarrow \frac{d^2y}{dv^2} + \frac{dy}{dv} - 12y = 0$$

This is of the form $ay'' + by' + cy = 0$, where y

is a function of v.

The char. eqn is

$$r^2 + r - 12r = 0$$

$$\Rightarrow (r+4)(r-3) = 0$$

$$\Rightarrow r_1 = -4 \quad \text{and} \quad r_2 = 3 \quad \text{(distinct roots)}$$

So $y = c_1 y_1 + c_2 y_2 = c_1 e^{-4v} + c_2 e^{3v}$

$$= c_1 e^{-4\ln x} + c_2 e^{3\ln x}$$

$$\Rightarrow y(x) = c_1 x^{-4} + c_2 x^3$$

This is the general solution of Eq(9).