2.3 Acceleration-Velocity Models

Resistance Proportional to Velocity

Example 1 Suppose that a body moves through a resisting medium with resistance proportional to its velocity v, so that $dv/dt = -kv$.
(a) Show that its velocity and position at time t are given by

$$v(t) = v_0 e^{-kt}$$

And

$$x(t) = x_0 + (\frac{v_0}{k}(1 - e^{-kt})).$$

(b) Conclude that the body travels only a finite distance, and find that distance.

ANS: (1) $\frac{dv}{dt} = -kv$ (sep.)

$\Rightarrow \int \frac{dv}{v} = -k\int dt$

$\Rightarrow \ln|v| = -kt + C$

$\Rightarrow \exp(\ln|v|) = \exp(-kt+C)$

$\Rightarrow v(t) = Ce^{-kt}$

Since $v(0) = v_0$,

$v_0 = Ce^{-0} = C$.

Thus $\boxed{v(t) = v_0 e^{-kt}}$

As $\frac{dx(t)}{dt} = v(t) = v_0 e^{-kt}$

$\Rightarrow \int dx(t) = v_0 \int e^{-kt} dt$

$\Rightarrow x(t) = v_0 \int e^{-kt} dt$

Let $u = -kt$, $du = -k\,dt$

$\Rightarrow dt = -\frac{1}{k} du$

Thus $x(t) = v_0 \int e^{u} (-\frac{du}{k})$

$= -\frac{v_0}{k} \int e^{u} du$

$\Rightarrow x(t) = -\frac{v_0}{k} e^{-kt} + C$

Since $x(0) = x_0$

$x(0) = -\frac{v_0}{k} e^{0} + C = x_0$

$\Rightarrow C = x_0 + \frac{v_0}{k}$

Thus $x(t) = -\frac{v_0}{k} e^{-kt} + x_0 + \frac{v_0}{k}$

$\Rightarrow \boxed{x(t) = x_0 + \frac{v_0}{k}(1 - e^{-kt})}$

(b) $t \to \infty$, $\Rightarrow e^{-kt} \to e^{-\infty} \to 0$

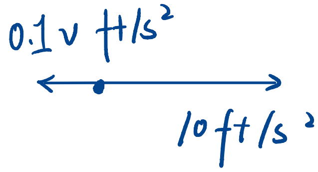
$\lim_{t\to\infty} x(t) = x_0 + \frac{v_0}{k}(1-0)$

$= \boxed{x_0 + \frac{v_0}{k}}$ finite distance

Example 2 Suppose that a car starts from rest, its engine providing an acceleration of $10ft/s^2$, while air resistance provides $0.1ft/s^2$ of deceleration for each foot per second of the car's velocity.

(a) Find the car's maximum possible (limiting) velocity.

(b) Find how long it takes the car to attain 90% of its limiting velocity, and how far it travels while doing so.



ANS: (a) (sep)

$$\frac{dv}{dt} = 10 - 0.1v, \quad v(0) = 0$$

$$\Rightarrow \int \frac{dv}{10-0.1v} = \int dt$$

Let $u = 10 - 0.1v$, $\Rightarrow du = -0.1dv$

$\Rightarrow dv = -10\,du$

So $-10\int \frac{du}{u} = \int dt$

$$\Rightarrow \ln|u| = -\frac{t}{10} + c$$

$$\Rightarrow \ln|10-0.1v| = -\frac{t}{10} + c$$

Since $v(0) = 0$.

$\ln 10 = c$

Thus $\ln|10-0.1v| = -\frac{t}{10} + \ln 10$

$$\Rightarrow \ln|10-0.1v| - \ln 10 = -\frac{t}{10}$$ $\ln x - \ln y = \ln\frac{x}{y}$

$$\Rightarrow \ln\left|\frac{10-0.1v}{10}\right| = -\frac{t}{10}$$

$$\Rightarrow \ln\left|1 - \frac{1}{100}v\right| = -\frac{t}{10}$$

$$\Rightarrow \quad 1-\frac{1}{100}v = e^{-\frac{t}{10}}$$

$$\Rightarrow \quad v(t) = 100\left(1-e^{-\frac{t}{10}}\right)$$

V_{max}

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 100\left(1-\cancel{e^{-\frac{t}{10}}}^{\,0}\right) = \boxed{100 \text{ ft/sec}}$$

(b) Let

$$v(t) = 90\% \ V_{max}$$
$$= 90 \text{ ft/sec}$$

$$\Rightarrow \quad 100\left(1-e^{-\frac{t}{10}}\right) = 90$$

$$\Rightarrow \quad 1-e^{-\frac{t}{10}} = 0.9$$

$$\Rightarrow \quad e^{-\frac{t}{10}} = 0.1$$

$$\Rightarrow \quad -\frac{t}{10} = \ln 0.1$$

$$\Rightarrow \quad t = -10\ln 0.1 \approx \boxed{23.059 \text{ s}}$$

We need to know $x(23.059) = ?$

$$\frac{dx}{dt} = v = 100\left(1-e^{-\frac{t}{10}}\right)$$

$$-100\int e^{-\frac{t}{10}}dt$$
$$= -100(-10)\int e^{-\frac{t}{10}} d\left(-\frac{t}{10}\right)$$
$$= 1000\, e^{-\frac{t}{10}} + C_1$$

$$\Rightarrow \quad x(t) = \int 100\left(1-e^{-\frac{t}{10}}\right)dt$$

$$\Rightarrow \quad x(t) = 100t - 100\int e^{-\frac{t}{10}}\,dt$$

$$\Rightarrow \quad x(t) = 100t + 1000\,e^{-\frac{t}{10}} + C_1$$

Since $x(0)=0$,

$$x(0) = 1000\,\cancel{e^{-0}}^{\,1} + C_1 = 0$$

$$\Rightarrow \quad C_1 = -1000$$

$$x(t) = 100t + 1000\,e^{-\frac{t}{10}} - 1000$$

Thus $\boxed{x(23.059) \approx 1402.59 \text{ ft}}$

Resistance Proportional to Square of Velocity and other cases

There are mathematical models with resistance proportional to other functions of velocities. For example, Page 96 and Question 6 on Page 100.