2.1 Population Models

Separable Equations and Partial Fraction Methods

Example 1 (review of Section 1.4) Separate variables and use partial fractions to solve the initial value problems. Sketch the graphs of several solutions of the given differential equation, and highlight the particular solution.

$$\frac{dx}{dt} = 1 - x^2, \quad x(0) = 3 \tag{1}$$

ANS: If $1-x^2 \neq 0$, we have

$$\int \frac{dx}{1-x^2} = \int dt \quad ①$$

Assume $\dfrac{1}{1-x^2} = \dfrac{1}{(1-x)(1+x)} = \dfrac{A}{1+x} + \dfrac{B}{1-x} = \dfrac{(B-A)x + A + B}{1-x^2}$

Compare the coeff.

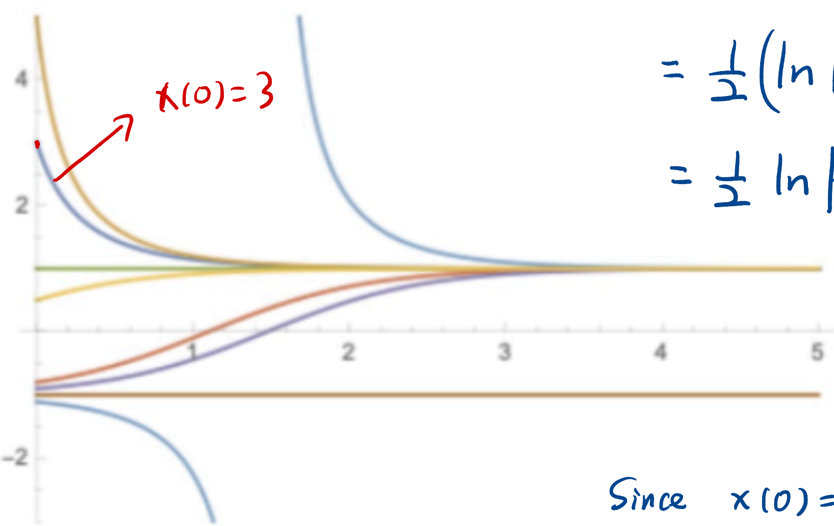
$$\begin{cases} B-A=0 \\ A+B=1 \end{cases} \Rightarrow A=B=\frac{1}{2}$$

$$\int \frac{1}{1-x}dx = -\int \frac{1}{1-x}d(1-x) = -\ln|1-x|$$

So ① becomes $t + C_1 = \int dt = \int \frac{dx}{1-x^2} = \frac{1}{2}\int \left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$

$$= \frac{1}{2}\left(\ln|1+x| - \ln|1-x|\right)$$

$$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = t + C_1$$



$$\Rightarrow \ln\left|\frac{1+x}{1-x}\right| = 2t + 2C_1$$

$$\Rightarrow \frac{1+x}{1-x} = Ce^{2t} \quad \text{(general solution)}$$

Since $x(0)=3$, $\frac{1+3}{1-3} = Ce^{2\cdot 0} = C$

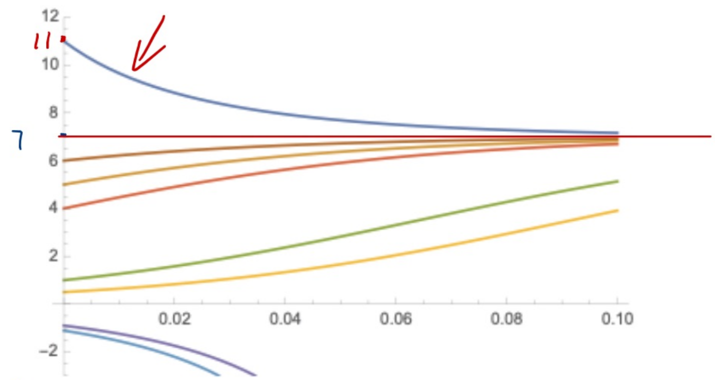
$\Rightarrow C = -2$. So $\frac{1+x}{1-x} = -2e^{2t}$

$$\Rightarrow x = \frac{2e^{2t}+1}{2e^{2t}-1} \quad \text{(particular sol.)}$$

If $1-x^2=0 \Rightarrow x(t) = \pm 1$ (singular sol.)

Exercise 2 (See the solution in the filled-in notes). Separate variables and use partial fractions to solve the initial value problems.
Sketch the graphs of several solutions of the given differential equation, and highlight the particular solution.

$$\frac{dx}{dt} = 4x(7-x), \quad x(0) = 11$$



ANS: If $4x(7-x) \neq 0$,

$$\boxed{\int \frac{dx}{x(7-x)}} = \int 4\,dt$$

$$= \frac{1}{7}\int \left(\frac{1}{x} + \frac{1}{7-x}\right) dx = 4t + C$$

Assume $\frac{1}{x(7-x)} = \frac{A}{x} + \frac{B}{7-x} = \frac{7A - Ax + Bx}{x(7-x)} = \frac{(B-A)x + 7A}{x(7-x)}$

$$\Rightarrow \begin{cases} B - A = 0 \\ 7A = 1 \end{cases} \Rightarrow A = B = \frac{1}{7}$$

$$\Rightarrow \frac{1}{7} \ln\left|\frac{x}{7-x}\right| dx = 4t + C$$

$$\Rightarrow \frac{x}{7-x} = Ce^{28t}$$

Since $x(0) = 11$,

$$\frac{11}{7-11} = C$$

$$\Rightarrow C = -\frac{11}{4}.$$

So $\frac{x}{7-x} = -\frac{11}{4} e^{28t}$

$$\Rightarrow x(t) = \frac{77}{11 - 4e^{-28t}}$$

Review: Exponential Growth Model

- An Example:
  - I mixed a cup of suger, water, ginger bugs and ginger into a jar.

  - One day later (a video)
- Earlier we used the exponential differential equation

$$\frac{dP}{dt} = kP \tag{3}$$

with solution

$$P(t) = P_0 e^{kt} \tag{4}$$

to model natural population growth.

- This assumed that the birth and death rates were constant.
- Now we consider a more general population model that allows for nonconstant birth and death rates.

Variable Birth and Death Rates

- We define the birth rate function $\beta(t)$ as the number of births per unit of population per unit of time at time t.
- Similary, the death rate function $\delta(t)$ is the number of deaths per unit of population per unit of time at time t.
- Over the time interval $[t, t+\Delta t]$ there are then roughl $\beta(t) \cdot P(t) \cdot \Delta t$ births and $\delta(t) \cdot P(t) \cdot \Delta t$ deaths
- Thus the change in population over this time interval is

$$\Delta P = \{ \text{ births } \} - \{ \text{ deaths } \} \approx \beta(t) \cdot P(t) \cdot \Delta t - \delta(t) \cdot P(t) \cdot \Delta t \tag{5}$$

- Dividing by Δt gives

$$\frac{\Delta P}{\Delta t} \approx [\beta(t) - \delta(t)] P(t) \tag{6}$$

- Taking the limit as $\Delta t \to 0$ gives the general population equation

$$\boxed{\frac{dP}{dt} = (\beta(t) - \delta(t)) P} \tag{7}$$

- In the event that β and δ are constant, this equation reduces to the natural growth equation with $k = \beta - \delta$.
- But it also includes the possibility that β and δ vary with t.

The Logistic Equation

Decreasing Birth Rate

- We often observe that the birth rate of a population decreases as the population itself grows.
- One way to model this is to assume that the birth rate β is a linear decreasing function of the population size P, then

$$\beta = \beta_0 - \beta_1 P \tag{8}$$

where β_0 and β_1 are positive constants.

- If the death rate $\delta = \delta_0$ remains constant, then our general population equation becomes

$$\frac{dP}{dt} = (\beta - \delta)P = (\beta_0 - \beta_1 P - \delta_0)P \tag{9}$$

- We can rewrite this as

$$\boxed{\frac{dP}{dt} = aP - bP^2} = \overset{k}{\overset{\shortparallel}{b}}\, P \left(\overset{M}{\overset{\shortparallel}{\frac{a}{b}}} - P \right) \tag{10}$$

where $a = \beta_0 - \delta_0$ and $b = \beta_1$.

- If the coefficients a and b are both positive, then this equation is called the logistic equation.
- It is useful to rewrite the logistic equation in the form

$$\boxed{\frac{dP}{dt} = kP(M - P)} \tag{11}$$

where $k = b$ and $M = a/b$ are constants.

Limiting Populations and Carrying Capacity

- In Section 1.4. The exponential differential equation has a general solution $P(t) = P_0 e^{kt}$
- It follows that

$$\lim_{t \to +\infty} P(t) = +\infty \tag{12}$$

- This means that the population grows without bound in a naturally growing population model.
- **Question:** If a population satisfies the logistic equation, what can we say about the population in the long-term?

Example 3. Show that the solution of the logistic initial value problem

$$\frac{dP}{dt} = kP(M-P), \quad P(0) = P_0 \tag{13}$$

is

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}} \tag{14}$$

Make it clear how your derivation depends on whether $0 < P_0 < M$ or $P_0 > M$.

ANS: If $P \neq 0$ and $M-P \neq 0$, then

$$\int \frac{dP}{P(M-P)} = \int k\,dt$$

Assume

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P} = \frac{AM - AP + BP}{P(M-P)} = \frac{(B-A)P + AM}{P(M-P)}$$

Compare the coeff.

$$\begin{cases} B-A=0 \\ AM = 1 \end{cases} \Rightarrow A = B = \frac{1}{M}$$

$$\Rightarrow \frac{1}{M}\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k\,dt$$

$$\Rightarrow \frac{1}{M}\left(\ln|P| - \ln|M-P|\right) = kt + C_1$$

$$\Rightarrow \ln\left|\frac{P}{M-P}\right| = kMt + C_2$$

$$\Rightarrow \frac{P}{M-P} = Ce^{kMt}$$

Since $P(0) = P_0$, $\frac{P_0}{M-P_0} = C$. we have

$$\frac{P}{M-P} = \frac{P_0}{M-P_0} e^{kMt}$$

$$\Rightarrow P(M-P_0) = P_0(M-P)e^{kMt}$$

$$\Rightarrow P(M-P_0) + PP_0e^{kMt} = P_0Me^{kMt}$$

$$\Rightarrow P[(M-P_0) + P_0e^{kMt}] = P_0Me^{kMt}$$

$$\Rightarrow P(t) = \frac{(P_0Me^{kMt})e^{-kMt}}{((M-P_0)+P_0e^{kMt})e^{-kMt}}$$

$$\Rightarrow \boxed{P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}}$$

- If $P_0 = M$, we have $P(t) = \frac{MP_0}{P_0 + 0} = M$ (our solution reduces to the constant valued "equilibrium population")
- If $0 < P_0 < M$, $\frac{dP}{dt} = kP(M-P) > 0$ near P_0 and

$$P(t) = \frac{MP_0}{P_0 + \{\text{pos. number}\}} < \frac{MP_0}{P_0} = M$$

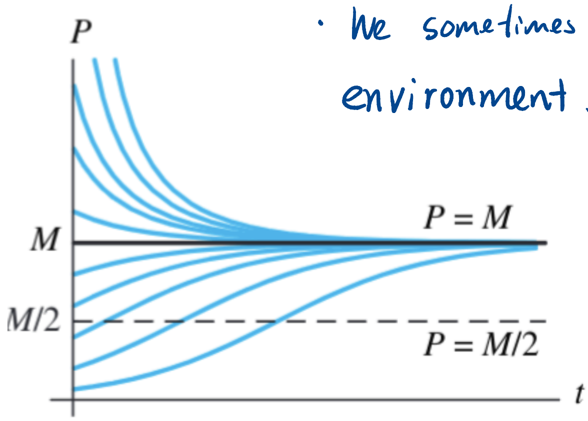
- If $P_0 > M$, $\frac{dP}{dt} = kP(M-P) < 0$ near P_0 and

$$P(t) = \frac{MP_0}{P_0 + \{\text{neg. number}\}} > \frac{MP_0}{P_0} = M$$

- In either case we find

$$\lim_{t\to\infty} P(t) = \frac{MP_0}{P_0 + 0} = M$$

- It approaches the finite limiting population M as $t \to \infty$
- We sometimes call M the carrying capacity of the environment.



Example 4

- Consider a population $P(t)$ satisfying the logistic equation

$$\frac{dP}{dt} = aP - bP^2 \tag{15}$$

where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur.

- If the initial population is $P(0) = P_0$, and B_0 births per month and D_0 deaths per month occurring at time $t = 0$

$\frac{a}{b}$

- Show that the limiting population is $M = B_0 P_0 / D_0$.

ANS: We know $M = \frac{a}{b}$ since

$$\frac{dP}{dt} = aP - bP^2 = bP\left(\frac{a}{b} - P\right) = bP(M - P)$$

At $t = 0$

$$P(0) = P_0$$

$$B_0 = aP_0 \Rightarrow a = \frac{B_0}{P_0}$$

$$D_0 = bP_0^2 \Rightarrow b = \frac{D_0}{P_0^2}$$

So we have

$$M = \frac{a}{b} = \frac{B_0/P_0}{D_0/P_0^2} = \frac{B_0 P_0}{D_0}$$

which proves the result.

Example 5

- Consider a rabbit population $P(t)$ satisfying the logistic equation as in **Example 4** .
- If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$
- How many months does it take for $P(t)$ to reach 95% of the limiting population M?

ANS: First let's find the diff. eqn. for $P(t)$.

Recall
$$\frac{dP}{dt} = aP - bP^2 = bP\left(\frac{a}{b} - P\right) = kP(M - P)$$

and $k = b$, $M = \frac{a}{b}$

At $t = 0$,
$$P_0 = 120$$
$$B_0 = 8 = aP_0$$
$$D_0 = 6 = bP_0^2$$
$$\Rightarrow b = \frac{D_0}{P_0^2}$$
from example 3.

From example 4,
$$M = \frac{B_0 P_0}{D_0} = \frac{8 \cdot 120}{6} = 160$$
$$k = b = \frac{D_0}{P_0^2} = \frac{6}{120^2} = \frac{1}{2400}$$

So we have
$$\frac{dP}{dt} = \frac{1}{2400} P(160 - P)$$

By Example 3, we know
$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$
$$= \frac{160 \cdot 120}{120 + (160 - 120)e^{-\frac{1}{2400} \cdot 160t}}$$
$$\Rightarrow P(t) = \frac{19200}{120 + 40e^{-\frac{1}{15}t}}$$

The question asks us what is t when $P(t) = 0.95M$

So we solve

$$P(t) = \frac{480}{3 + e^{-\frac{1}{5}t}} = 0.95 \cdot 160$$

$\Rightarrow$ $t \approx 27.69$ months