Review:

In 1.1-1.5, we talked about

- $\dfrac{dy}{dx} = f(x) \implies y = \displaystyle\int f(x)dx + C$
- **Separable Equation** $\dfrac{dy}{dx} = g(x)k(y)$

  If $k(y) \neq 0$, $\displaystyle\int \frac{dy}{k(y)} = \int g(x)dx$

  Also we need to check if $k(y) = 0$ is a solution.
- **Linear First Order Equation** $\dfrac{dy}{dx} + P(x)y = Q(x)$ (*)
  1. Compute $\rho(x) = e^{\int P(x)dx}$ (integrating factor).
  2. Multiply both sides of (*) by $\rho(x)$
  3. $\text{LHS} = D_x(\rho(x)y(x))$
  4. Integrate both sides, $\rho(x)y(x) = \displaystyle\int \rho(x)Q(x)dx + C$ and solve for y.

Outline of Section 1.6

1. **Substitution Method**
   - Equation: $\dfrac{dy}{dx} = F(ax + by + c)$
   - Homogeneous Equations: $\dfrac{dy}{dx} = F(\dfrac{y}{x})$
   - Bernoulli Equations: $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$
   - Reducible Second-order Equations:

     $$F(x, y, y'y'') = 0$$

     with either y or x is missing.
2. **Exact Equations**
   - What is an exact equation?
   - How to check an equation is exact?
   - How to solve an exact equation?

Part 1 Substitution Method

Often a substitution can be used to transform a given differential equation into one that we already know how to solve. For example,

The differential equation of the form

$$\frac{dy}{dx} = F(ax + by + c) \tag{1}$$

can be transformed into a separable equation by use of the substitution $v = ax + by + c$.

Example 1 Find a general solution of the differential equation

$$\frac{dy}{dx} = (9x + y)^2 \quad ① \tag{2}$$

ANS: Let $v = 9x + y$. Our goal is to transform ① into an eqn in term of $\frac{dv}{dx}$. Then $\frac{dv}{dx} = 9 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 9$.

Substitute them into ①, then

$$\frac{dv}{dx} - 9 = v^2$$

$$\Rightarrow \quad \frac{dv}{dx} = v^2 + 9 \quad \text{(separable)}$$

Separating variable and integrating both sides,

$$\int \frac{dv}{v^2+9} = \int dx \qquad \frac{1}{9}\int \frac{dv}{\left(\frac{v}{3}\right)^2 + 1} \quad \text{(u-subs)}$$

Let $u = \frac{v}{3}$, $\Rightarrow du = \frac{1}{3} dv$

$\Rightarrow \frac{1}{3}\int \frac{du}{u^2+1} = \frac{1}{3}\tan^{-1} u$

back subs. $u = \frac{v}{3}$: $\frac{1}{3}\tan^{-1}\frac{v}{3}$

$$\Rightarrow \frac{1}{3}\tan^{-1}\frac{v}{3} = x + C_1$$

$$\Rightarrow \tan^{-1}\frac{v}{3} = 3x + C$$

$$\Rightarrow \frac{v}{3} = \tan(3x + C)$$

$$\Rightarrow v = 3\tan(3x + C)$$

Back substitute $v = 9x + y$.

$$9x + y = 3\tan(3x + C)$$

$$\Rightarrow \boxed{y = 3\tan(3x + C) - 9x}$$

- Homogeneous Equations

  A homogeneous first-order differential equation is one that can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right). \tag{3}$$

The substitution $v = \frac{y}{x}$, that is, $y = vx$ leads to

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{4}$$

by the product rule.

The given equation $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ then becomes

$$v + x\frac{dv}{dx} = F(v) \implies x\frac{dv}{dx} = F(v) - v \tag{5}$$

which is a separable differential equation for v as a function of x.

Example 2 Find a general solution of the differential equation

$$\underset{x^2y^0\quad x^0y^2}{(x^2 - y^2)}\frac{dy}{dx} = \underset{x^1y^1}{2xy} \quad \cdots ① \tag{6}$$

ANS: If $x^2-y^2 \neq 0$, $x \neq 0$, we can rewrite ① as

$$\frac{dy}{dx} = \frac{(2xy)/x^2}{(x^2-y^2)/x^2} = \frac{2\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2} \quad \left(= F\left(\frac{y}{x}\right)\right) \quad ②$$

Let $v = \frac{y}{x}$, then $y = vx$. $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute them into ②,

$$v + x\cdot\frac{dv}{dx} = \frac{2v}{1-v^2}$$

$$\Rightarrow x\cdot\frac{dv}{dx} = \left(\frac{2v}{1-v^2} - v = \frac{2v - v + v^3}{1-v^2} =\right) \frac{v^3+v}{1-v^2} \quad \text{(separable)}$$

Separating variables and integrate.

$$\boxed{\int \frac{1-v^2}{v^3+v}\,dv} = \int \frac{1}{x}\,dx \qquad ③$$

To solve

$$\int \frac{1-v^2}{v^3+v}\,dv = \int \frac{1-v^2}{v(v^2+1)}\,dv$$

Recall the partial fraction method.

Assume $\frac{1-v^2}{v(v^2+1)} = \frac{A}{v} + \frac{Bv+C}{v^2+1}$

$$= \frac{Av^2+A+Bv^2+Cv}{v(v^2+1)}$$

$$\Rightarrow \quad \frac{-v^2+1}{v(v^2+1)} = \frac{(A+B)v^2+Cv+A}{v(v^2+1)}$$

Comparing the coefficients, we have

$$\begin{cases} A+B=-1 \\ C=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-2 \\ C=0 \end{cases}$$

Thus

$$\frac{1-v^2}{v(v^2+1)} = \frac{1}{v} - \frac{2v}{v^2+1}$$

u-subs. Let $u = v^2+1$, $\int \frac{du}{u} = \ln|u|$

$$\int \frac{d(v^2+1)}{v^2+1} = \ln|v^2+1|$$

So $$\int \frac{1-v^2}{v(v^2+1)}\,dv = \int\left(\frac{1}{v} - \frac{2v}{v^2+1}\right)dv = \ln|v| - \int \frac{2v}{v^2+1}\,dv$$

$$= \ln|v| - \ln|v^2+1| = \ln\left|\frac{v}{v^2+1}\right|$$

So ③ becomes

$$\ln\left|\frac{v}{v^2+1}\right| = \ln|x| + C_1$$

$\xRightarrow{\text{Take exp}}$

$$\frac{v}{v^2+1} = Cx$$

Substitute $v = \frac{y}{x}$ back.

$$\frac{\left(\frac{y}{x}\right)x^2}{\left(\left(\frac{y}{x}\right)^2+1\right)x^2} = Cx$$

$$\Rightarrow \quad \frac{\cancel{x}y}{y^2+x^2} = C\cancel{x}$$

$$\Rightarrow \quad \boxed{y = C(x^2+y^2)}$$

The following table indicates some simple partial fractions which can be associated with various rational functions:

Form of the rational function	Form of the partial fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{\mathrm{A}}{x-a}+\frac{\mathrm{B}}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{\mathrm{A}}{x-a}+\frac{\mathrm{B}}{x-b}+\frac{\mathrm{C}}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{\mathrm{A}}{x-a}+\frac{\mathrm{B}}{(x-a)^2}+\frac{\mathrm{C}}{x-b}$
$\frac{px^2+qx+r}{(x-a)\left(x^2+bx+c\right)}$ where x^2+bx+c cannot be factorised further	$\frac{\mathrm{A}}{x-a}+\frac{\mathrm{B}x+\mathrm{C}}{x^2+bx+c}$

Exercise 3 (Check the answer from the filled-in notes)
Find a general solution of the differential equation

$$x\frac{dy}{dx} = y + \sqrt{x^2 + y^2} \quad ①$$

ANS: If $x > 0$, we can divide both sides of ① by x.

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{x^2}{x^2} + \left(\frac{y}{x}\right)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} \quad ②$$ (homogeneous equation)

Let $v = \frac{y}{x}$, then $y = vx$ and $\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v$

Subsitute them into ②, we have

$$\frac{dv}{dx} \cdot x + v = v + \sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{dx} \cdot x = \sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \frac{1}{x}\, dx$$

By checking an integral table, we know

$$\ln\left(v + \sqrt{v^2+1}\right) = \ln|x| + C$$

$$\Rightarrow \quad v + \sqrt{v^2+1} + Cx$$

Back-subsituting, $v = \frac{y}{x}$, we have

$$\frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 + 1} = Cx$$

$$\Rightarrow \quad \boxed{y + \sqrt{y^2 + x^2} = Cx^2}$$

Bernoulli Equations

A first-order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{1}$$

is called a Bernoulli equation.

If either $\mathrm{n} = 0$ or $\mathrm{n} = 1$, Eq. (1) is linear.

In our homework, we need to show the substitution

$$v = y^{1-n} \tag{8}$$

transforms Eq. (1) into the linear first-order equation

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x) \tag{9}$$

Example 4 Find a general solution of the differential equation

$$x^2\frac{dy}{dx} + 2xy = 5y^4 \quad ① \tag{10}$$

ANS: If $x \neq 0$, We divide both sides of ① by x^2

$$\frac{dy}{dx} + \underbrace{\frac{2}{x}}_{P(x)} y = \underbrace{\frac{5}{x^2}}_{Q(x)} y^{\underbrace{4}_{n=4}} \quad \cdots \quad ②$$

This is a Bernouli equation. We let

$$V = y^{1-n} = y^{1-4} = y^{-3}$$

Then $y = V^{-\frac{1}{3}}$ $\left((V)^{-\frac{1}{3}} = (y^{-3})^{-\frac{1}{3}} = y \right)$

Taking diff. both sides

$$\frac{dy}{dx} = -\frac{1}{3} V^{-\frac{4}{3}} \frac{dv}{dx}$$

Substitute $y = V^{-\frac{1}{3}}$, $\frac{dy}{dx} = -\frac{1}{3} V^{-\frac{4}{3}} \frac{dv}{dx}$ into ②.

$$-\frac{1}{3} V^{-\frac{4}{3}} \frac{dv}{dx} + \frac{2}{x} \cdot V^{-\frac{1}{3}} = \frac{5}{x^2} V^{-\frac{4}{3}}$$

We multiply both sides by $-3v^{\frac{4}{3}}$, we have

$$-\frac{1}{3}v^{-\frac{4}{3}}(-3v^{\frac{4}{3}})\frac{dv}{dx} + \frac{2}{x}\cdot v^{-\frac{1}{3}}\cdot(-3v^{\frac{4}{3}}) = \frac{5}{x^2}v^{-\frac{4}{3}}\cdot(-3v^{\frac{4}{3}})$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{6}{x}v = -\frac{15}{x^2} \quad (\text{Linear 1st order}) \quad ③$$

· An integrating factor

$$\rho(x) = e^{\int -\frac{6}{x}dx} = e^{\ln|x|^{-6}} = x^{-6} = \frac{1}{x^6}$$

· Multiply both sides of ③ by $\rho(x)$

$$\frac{1}{x^6}\cdot\frac{dv}{dx} - \frac{6}{x}\cdot\frac{1}{x^6}v = -\frac{15}{x^8}$$

· Note

$$LHS = D_x(\rho(x)\,v(x)) = D_x\left(\frac{1}{x^6}\cdot v(x)\right)$$

· Integrate both sides.

$$\frac{1}{x^6}v(x) = -\int\frac{15}{x^8}dx = -15\int x^{-8}dx$$

$$= -\frac{15}{-7}x^{-7} + C$$

$$\Rightarrow \quad v(x) = \frac{15}{7}\cdot\frac{1}{x} + C\cdot x^6$$

· Back substitute $v = y^{-3}$, we have

$$(y^{-3})^{-\frac{1}{3}} = \left(\frac{15}{7}\cdot\frac{1}{x} + C\cdot x^6\right)^{-\frac{1}{3}}$$

$$\Rightarrow \quad \boxed{y = \left(\frac{15}{7}\cdot\frac{1}{x} + C\cdot x^6\right)^{-\frac{1}{3}}}$$

Reducible Second-Order Equations
Read Page $67-69$ in our textbook.
A second-order differential equation has the general form

$$F\left(x, y, y', y''\right)=0 \tag{2}$$

It may be that the dependent variable y or the independent variable x is missing from a second-order equation.

Case 1. Variable y Missing

- If y is missing, then our equation takes the form

$$F\left(x, y', y''\right)=0 \tag{11}$$

- Then the substitution

$$p=y'=\frac{dy}{dx}, \quad y''=\frac{dp}{dx} \tag{12}$$

results in the first-order differential equation

$$F\left(x, p, p'\right)=0 \tag{13}$$

Example 5 Find a general solution of the reducible second-order differential equation

$$x y''=y' \tag{14}$$

ANS: Let $p=y'=\frac{dy}{dx}$, then $y''=\frac{dp}{dx}$ $(=p')$

Then subs. them into (14), then

$$x\frac{dp}{dx}=p \quad (\text{sep.})$$

If $p\neq 0$, then $\displaystyle\int\frac{dp}{p}=\int\frac{dx}{x}$

$$\Rightarrow \ln|p|=\ln|x|+c$$

$$\Rightarrow e^{\ln|p|}=e^{\ln|x|+c}=e^c e^{\ln|x|}=c' e^{\ln|x|}$$

$$\Rightarrow p=c_1 x$$

Now $p=\frac{dy}{dx}=c_1x \Rightarrow y=\int c_1 x\,dx=\frac{1}{2}c_1x^2+c_2$

Case 2. Variable x Missing

- If x is missing, then our equation takes the form

$$F(y, y', y'') = 0$$

- Then the substitution

$$p = y' = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

results in the first-order differential equation

$$F\left(y, p, p\frac{dp}{dy}\right) = 0$$

for p as a function of y.

Exercise 6 (Check the answer from the filled-in notes.) Find a general solution of the reducible second-order differential equation

$$yy'' + (y')^2 = yy' \qquad (1)$$

ANS: Let $p = y' = \frac{dy}{dx}$, then $y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \cdot \frac{dp}{dy}$

Plug them into eqn (1), we get.

$$y \cdot p \cdot \frac{dp}{dy} + p^2 = y \cdot p$$

If $p \neq 0$, then

$$y \cdot \frac{dp}{dy} + p = y \quad --- (*)$$

If $y \neq 0$, then

$$\frac{dp}{dy} + \frac{1}{y} \cdot p = 1. \qquad (2)$$

(A linear first order eqn, where p is a function of y)

Note: In fact, we observe that LHS of ⊛ is the derivative of py w.r.t. y, which means $LHS = y\frac{dp}{dy} + p = \frac{\partial(py)}{\partial y}$. So we can integrate both sides of ⊛ and get $py = \int y\,dy \Rightarrow py = \frac{1}{2}y^2 + C_1$

You can use the method from §1.5 to check we get the same answer from ②

So we have $py = \frac{1}{2}y^2 + C_1 \Rightarrow p = \frac{y^2 + C_1}{2y}$

Since $p = \frac{dy}{dx} = \frac{y^2 + C_1}{2y} \Rightarrow \int \frac{2y\,dy}{y^2 + C_1} = \int dx$

$$\Rightarrow \int \frac{d(y^2 + C_1)}{y^2 + C_1} = \int dx \Rightarrow \ln|y^2 + C_1| = x + C_2 \Rightarrow \boxed{y^2 = C_2 e^x + C_1}$$

Part 2 Exact Equations

Consider $F(x, y(x)) = C$, which implicitly defines y as a function of x.

For example, $F(x,y) = x^3 + 2xy^2 + 2y^3 = C$.

Differentiating both sides with respect to x, then we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0 \tag{19}$$

Let $M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$. We can rewrite it in differential form

$$M(x,y)dx + N(x,y)dy = 0. \tag{3}$$

Then $F(x,y) = C$ is a solution of Eq (3).

For example, differentiating both sides of $F(x,y) = x^3 + 2xy^2 + 2y^3 = C$, we have

$$(3x^2 + 2y^2) + (4xy + 6y^2)\frac{dy}{dx} = 0, \tag{20}$$

which can be rewrite as

$$(3x^2 + 2y^2)dx + (4xy + 6y^2)dy = 0, \tag{21}$$

Note $F(x,y) = x^3 + 2xy^2 + 2y^3 = C$ is a solution to the above equation.

Defintion. Exact Equation

Generally, consider the following equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{4}$$

If there exists a function $F(x,y)$ such that

$$F_x = \frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N = F_y \tag{22}$$

then the equation

$$F(x,y) = C \tag{23}$$

is an implicit general solution of Eq. (4). We call such Eq. (4) an exact equation.

🤔 How can we check whether the eqn (4) is exact ?

If F_{xy} & F_{yx} are continuous on open set in the xy-plane. Then $F_{xy} = F_{yx}$

$$F_{xy} = \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} = F_{yx}$$

It turns out $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is necessary & sufficient for exactness

THEOREM 1 Criterion for Exactness

Suppose that the functions

$$M(x,y) \quad \text{and} \quad N(x,y) \tag{24}$$

are continuous and have continuous first-partial order derivatives in the open rectangle

$$R: a < x < b, c < y < d \tag{25}$$

Then the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{5}$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{6}$$

at each point of R.

Example 7 Verify that the given differential equation is exact; then solve it.

$$\left(2xy^2 + 3x^2\right)dx + \left(2x^2y + 4y^3\right)dy = 0 \tag{26}$$

ANS: Let $M(x,y) = 2xy^2 + 3x^2$

$N(x,y) = 2x^2y + 4y^3$

Then $\frac{\partial M}{\partial y} = \frac{\partial(2xy^2 + 3x^2)}{\partial y} = 4xy$

$\qquad \parallel \qquad\qquad\qquad\qquad\qquad \parallel$

$\frac{\partial N}{\partial x} = \frac{\partial(2x^2y + 4y^3)}{\partial x} = 4xy$

By Thm 1, the given eqn is exact.

Then by the definition of exact eqn.

There exist $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) = 2xy^2 + 3x^2$$

$$\Rightarrow F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \int (2xy^2 + 3x^2)\,dx$$

$$\Rightarrow \quad F(x,y) = x^2y^2 + x^3 + g(y)$$

We diff. both sides in terms of y.

$$N(x,y) \underset{\substack{\uparrow \\ \text{by def}}}{=} \frac{\partial F}{\partial y} = 2x^2y + 0 + \frac{dg(y)}{dy}$$

$$\Rightarrow \cancel{2x^2y} + 4y^3 = \cancel{2x^2y} + \frac{dg(y)}{dy}$$

$$\Rightarrow \frac{dg(y)}{dy} = 4y^3$$

$$\Rightarrow \quad g(y) = \int \frac{dg(y)}{dy}\,dy = \int 4y^3\,dy = y^4$$

So $\underline{F(x,y) = x^2y^2 + x^3 + y^4 = C}$ is a general solution