1.5 Linear First-Order Equations

An example

Example 1 Find a general solution to the differential equation

$$\frac{dy}{dx} = 2xy \quad (y > 0) \tag{1}$$

ANS: Method 1 Notice that this is a separable diff eqn. So

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\Rightarrow \boxed{\ln y = x^2 + C}$$

Method 2 Rather than dividing both sides by y, we can multiply both sides of (1) by $\frac{1}{y}$.

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2x$$

$D_x(\ln y) = \frac{1}{y} \cdot \frac{dy}{dx}$ (chain rule)

We can recognize each side of the eqn as a derivative. i.e.

$$D_x(\ln y) = D_x(x^2)$$

Then integrating both sides gives us

$$\boxed{\ln y = x^2 + C}$$

In general, an integrating factor for a diff. eqn. is a function $\rho(x, y)$ with the property that multiplying each side of the eqn by $\rho(x,y)$ allows each side to be recognizable as derivative. For example, $\rho(x,y) = \frac{1}{y}$ is an integrating factor for this example.

Linear First-order Equations

A **linear first-order equation** is a differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{2}$$

where the coefficient functions $P(x)$ and $Q(x)$ are continuous on some interval on the x-axis.

- This equation can always be solved using the integrating factor

$$\rho(x) = e^{\int P(x)dx} \tag{3}$$

Rmk: No constant of the integration is needed when finding the integrating factor $\rho(x)$. Since replacing $\int P(x)\,dx$ with $\int P(x)dx + c$ leads to $\rho(x) = e^{c + \int P(x)dx} = e^{c} \cdot e^{\int P(x)dx}$

- Multiplying by $\rho(x)$ gives

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx} \tag{4}$$

- The left-hand side is now the derivative of the product

$$y(x) \cdot e^{\int P(x)dx} \tag{5}$$

Check: $D_x\left(y(x) \cdot e^{\int P(x)\,dx}\right) = \frac{dy}{dx} \cdot e^{\int P(x)\,dx} + y \cdot e^{\int P(x)dx} \cdot P(x)$

(product rule + chain rule)

- So we can rewrite our equation as

$$D_x\left[y(x) \cdot e^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx} \tag{6}$$

- Integrating both sides gives

$$y(x)e^{\int P(x)dx} = \int \left(Q(x)e^{\int P(x)dx}\right)dx + C \tag{7}$$

- Finally, solving for $y(x)$ gives

$$y(x) = e^{-\int P(x)dx}\left[\int \left(Q(x)e^{\int P(x)dx}\right)dx + C\right] \tag{8}$$

- **Note:** This formula is not to be memorized, but rather illustrates a general method that can be applied in specific cases.

We summarize the steps of the method as follows:

Method of Solution of Linear First-Order Equations

Rmk: We need to make sure we have the eqn in this form. See example 2 below

$$\frac{dy}{dx}+P(x)y=Q(x) \tag{9}$$

Step 1. Compute the integrating factor $\rho(x)=e^{\int P(x)dx}$.

Step 2. Multiply both sides of the differential equation by $\rho(x)$.

Step 3. Next, recognize the left-hand side of the resulting equation as the derivative of a product:

$$D_x[\rho(x)y(x)]=\rho(x)Q(x) \tag{10}$$

Step 4. Finally, integrate this equation,

$$\rho(x)y(x)=\int\rho(x)Q(x)dx+C \tag{11}$$

then solve for $y(x)$ to obtain the general solution of the original differential equation.

Example 2 Find a general solution to the differential equation

$$xy'=3y+x^4\cos x,\quad y(2\pi)=0 \tag{12}$$

ANS: We first write (12) in the form of $\frac{dy}{dx}+P(x)y=Q(x)$.

If $x\neq 0$, we can rewrite (12) as

$$\frac{dy}{dx}\underbrace{-\frac{3}{x}}_{P(x)}y=\underbrace{x^3\cos x}_{Q(x)} \quad (*)$$

Step 1. An integrating factor $\rho(x)=e^{\int P(x)dx}=e^{\int -\frac{3}{x}dx}=e^{-3\ln|x|}$

$=|x|^{-3}$. If $x>0$, $\rho(x)=x^{-3}$. If $x<-3$, $\rho(x)=-x^{-3}$

Step 2. Multiply both sides by $\rho(x)=x^{-3}$

$$x^{-3}\frac{dy}{dx}-3x^{-4}y=\cos x$$

Step 3. Note LHS $=D_x(x^{-3}y)$ $(=D_x(\rho(x)y(x)))$

Step 4. Integrate both sides in terms of x.

$$x^{-3}y=\int\cos x\,dx+C=\sin x+C$$

$$\Rightarrow\quad y=x^3\sin x+x^3\cdot C$$

As $y(2\pi)=0$, $(2\pi)^3\cdot \cancel{\sin 2\pi}^{0} + (2\pi)^3\cdot c = 0 \Rightarrow c=0$

$$\boxed{y = x^3 \sin x}$$

Example 3 Solve the following differential equation by regarding y as the independent variable rather than x

$$\left(1-4xy^2\right)\frac{dy}{dx} = y^3 \Rightarrow y^3 dx = (1-4xy^2)\,dy \tag{13}$$

ANS: Regarding y as the independent variable, we write the eqn as

$$\frac{dx}{dy} = \frac{1}{y^3}(1-4xy^2) = \frac{1}{y^3} - 4\frac{x}{y}$$

$$\Rightarrow \frac{dx}{dy} + \underbrace{\frac{4}{y}}_{P(y)} x = \underbrace{\frac{1}{y^3}}_{Q(y)} \quad ①$$

(This is a linear first order eqn. where y is the independent variable. i.e.

$$\frac{dx}{dy} + P(y)x = Q(y)\text{)}$$

Step 1. An integrating factor $\rho(y) = e^{\int P(y)dy} = e^{\int \frac{4}{y}dy} = e^{4\ln|y|} = y^4$

Step 2. Multiply $\rho(y)$ on both sides of ①.

$$\frac{dx}{dy}\cdot y^4 + 4xy^3 = y$$

Step 3. Note that LHS $= D_y(x\cdot y^4)$ $(= D_y(x\cdot \rho(y)))$

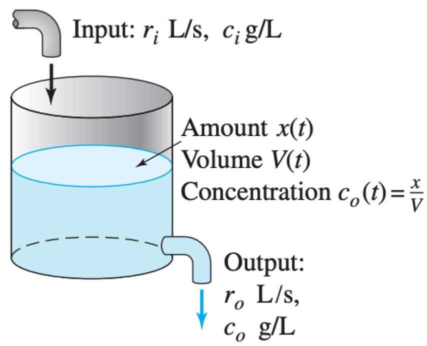
Step 4. So we integrate both sides in terms of y.

$$x\cdot y^4 = \int y\,dy = \frac{1}{2}y^2 + c$$

$$\Rightarrow \boxed{x(y) = \frac{1}{2}y^{-2} + c\cdot y^{-4}}$$

An Application of Linear First-Order Equations: Mixture Problems

- A tank containing a solution-a mixture of solute and solvent-has both inflow and outflow.
- Our goal is to find the amount $x(t)$ of solute at time t, given the initial amount x_0.
- Suppose that solution with a concentration of c_i grams of solute per liter of solution flows into the tank at the constant rate of r_i liters per second, and that the (mixed) solution in the tank flows out at the rate of r_o liters per second.



Analysis: Set up a differential equation for x

- We want to estimate the change Δx in x during the brief time interval $[t, t + \Delta t]$
- The amount of solute that flows into the tank during Δt seconds is $r_i c_i \Delta t$ grams.
- The amount that flows out of the tank is more complex because it depends upon the conceration $c_o(t) = \frac{x(t)}{V(t)}$ of solute in the solution at time t
- So the change Δx in the amount of solute is:

$$\Delta x = \{\text{grams input}\} - \{\text{grams output}\} \approx r_i c_i \Delta t - r_o c_o \Delta t$$

- Dividing by Δt, gives

$$\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o.$$

- Let $\Delta t \to 0$,

$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

- Note r_i, c_i, and r_o are constant. But $c_o(t) = \frac{x(t)}{V(t)}$.
- If $V_0 = V(0)$, then $V(t) = V_0 + (r_i - r_o) t$
  - So $c_o(t)$ is a constant when $r_i = r_o$.

Therefore,

$$\boxed{\frac{dx}{dt} = r_i c_i - r_o \frac{x(t)}{V(t)}, \text{ where } V(t) = V_0 + (r_i - r_o)t} \qquad (14)$$

Example 4 $x(0)=0$

A tank initially contains 240 gal of pure water. Brine containing $1/4\text{lb}$ of salt per gallon enters the tank at $2\text{gal}/\text{min}$, and the (perfectly mixed) solution leaves the tank at $4\text{gal}/\text{min}$; thus the tank is empty after exactly 2 h.

(a) Find the amount of salt in the tank after t minutes. $x(t)$

(b) What is the maximum amount of salt ever in the tank? X_{max}

ANS: $V(0)=V_0=240$ gal.

$r_i = 2\text{ gal/min}$, $c_i = 1/4 \text{ lb/gal}$

$r_o = 4\text{ gal/min}$, $c_o = \dfrac{x(t)}{V(t)}$, where $V(t) = V_0 + (r_i - r_o)t$

$= 240 - 2t$

We have

$$\frac{dx}{dt} = r_i c_i - r_o c_o, \text{ where } c_o = \frac{x(t)}{V(t)}$$

$$\Rightarrow \frac{dx}{dt} = 2\cdot\frac{1}{4} - 4\cdot\frac{x(t)}{240-2t}$$

$$\Rightarrow \frac{dx}{dt} + \underbrace{\frac{2}{120-t}}_{P(t)} x(t) = \underbrace{\frac{1}{2}}_{Q(t)} \quad ①$$

Step 1. An integrating factor $\rho(t) = e^{\int \frac{2}{120-t}dt} = e^{-2\ln(120-t)} = \dfrac{1}{(120-t)^2}$

Step 2. Multiply $\rho(t)$ on both sides of ①, we have

$$\frac{1}{(120-t)^2}\cdot\frac{dx}{dt} + \frac{2}{(120-t)^3}x = \frac{1}{2}\cdot\frac{1}{(120-t)^2} \quad ②$$

Step 3. Note $LHS = D_t(\rho(t)\cdot x(t)) = D_t\left(\frac{1}{(120-t)^2}\cdot x(t)\right)$

Step 4. Integrate both sides of ②, we have

$$\frac{1}{(120-t)^2}x(t) = \int \frac{1}{2}\cdot\frac{1}{(120-t)^2}dt.$$

$$= \int \frac{1}{2}(120-t)^{-2}dt$$

$$= -\int \frac{1}{2}(120-t)^{-2}\, d\,(120-t)$$ Also we can assume $u = 120-t$

$$= -\frac{1}{2}\cdot\frac{1}{1-2}\cdot(120-t)^{-1} + C$$

$$= \frac{1}{2}\cdot\frac{1}{120-t} + C$$

$$\Rightarrow \frac{1}{(120-t)^2}\,x(t) = \frac{1}{2}\cdot\frac{1}{120-t} + c$$

Since $x(0)=0$, $\frac{1}{120^2}\cdot 0 = \frac{1}{2}\cdot\frac{1}{120-0} + c$

$$\Rightarrow c = -\frac{1}{240}$$

Thus $\frac{1}{(120-t)^2}\,x(t) = \frac{1}{2}\cdot\frac{1}{120-t} - \frac{1}{240}$

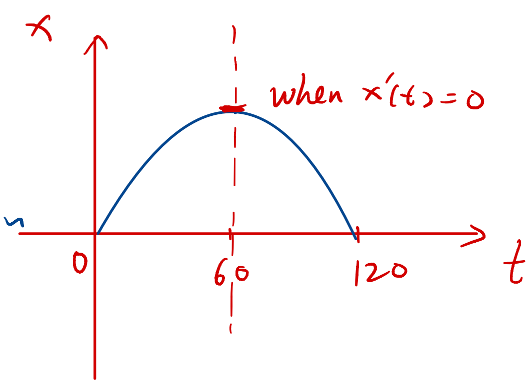
$$\Rightarrow x(t) = \frac{1}{2}\cdot(120-t) - \frac{1}{240}\cdot(120-t)^2$$

$$= -\frac{1}{240}(t-120)(\cancel{120}+t-\cancel{120})$$

$$\boxed{x(t) = -\frac{1}{240}\,t\,(t-120)}$$

(b). Method 1. From the graph, we know

x reaches x_{max} when $t=60$ min



Method 2. Note x reaches x_{max} when

$$x'(t) = -\frac{1}{120}t + \frac{1}{2} = 0 \quad\Rightarrow\quad t = 60\text{ min}$$

Example 5 A 120-gallon (gal) tank initially contains $90\mathrm{lb}$ of salt dissolved in 90 gal of water. Brine containing $2\mathrm{lb}/\mathrm{gal}$ of salt flows into the tank at the rate of $4\mathrm{gal}/\mathrm{min}$, and the well-stirred mixture flows out of the tank at the rate of 3 gal/min.

How much salt does the tank contain when it is full?

ANS: Let $x(t)$ be the amount of salt at time t.

$c_i = 2$ lb/gal

$r_i = 4$ gal/min

$r_o = 3$ gal/min

$c_o = \frac{x(t)}{V(t)} = \frac{x(t)}{90+t}$

$V_0 = V(0) = 90$ gal

$V(t) = V_0 + (r_i - r_o)t$

$= 90 + t$

$x(0) = 90$ lb

$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

$$\Rightarrow \frac{dx}{dt} = 4 \cdot 2 - 3 \cdot \frac{x(t)}{90+t}$$

$$\Rightarrow \frac{dx}{dt} + \underbrace{\frac{3}{90+t}}_{P(t)} x(t) = \underbrace{8}_{Q(t)} \qquad ①$$

- An integrating factor

$$\rho(t) = e^{\int \frac{3}{90+t} dt} = e^{3\ln(90+t)} = (90+t)^3$$

- Multiply both sides of ① by $\rho(t)$

$$(90+t)^3 \frac{dx}{dt} + 3 \cdot (90+t)^2 x = 8 \cdot (90+t)^3 \qquad ②$$

- $LHS = D_t(\rho(t)\, x(t)) = D_t((90+t)^3 \cdot x(t))$

- Integrate both sides of ②

$$(90+t)^3 \cdot x(t) = \int 8 \cdot (90+t)^3 dt = 2 \cdot (90+t)^4 + C$$

$$\Rightarrow (90+t)^3 \cdot x(t) = 2 \cdot (90+t)^4 + C$$

Since $x(0) = 90$, we have

$$90^3 \cdot 90 = 2 \cdot 90^4 + C$$

$$\Rightarrow C = -90^4$$

So $x(t) = 2 \cdot (90+t) - \dfrac{90^4}{(90+t)^3}$

The tank is full when

$$V(t) = 90 + t = 120$$

$$\Rightarrow t = 30 \text{ min}$$

$$x(30) = 2 \cdot (90+30) - \frac{90^4}{(90+30)^3}$$

$$\approx 202 \text{ lb}$$