1.4 Separable Equations and Applications

Recall in Section 1.2, we solved questions like

$$\frac{dy}{dx} = f(x) \tag{1}$$

The idea is *integrating both sides*. Can we apply the same idea for the following question?

Example 1. Find solutions of the differential equation $\dfrac{dy}{dx} = y\sin x$. ① $= k(y)\cdot f(x)$

ANS: If $y \neq 0$, we can divide both sides by y and multiply both sides by dx.

$$\frac{dy}{y} = \sin x \, dx$$

Integrate both sides, we have

$$\int \frac{dy}{y} = \int \sin x \, dx \Rightarrow \ln|y| = -\cos x + C_1$$

$$\Rightarrow e^{\ln|y|} = e^{-\cos x + C_1} \Rightarrow |y| = e^{C_1} \cdot e^{-\cos x}$$

$$\Rightarrow y = \underbrace{\pm e^{C_1}}_{\text{is a constant } C \neq 0} \cdot e^{-\cos x} = Ce^{-\cos x} \quad (C \neq 0)$$

$$\Rightarrow \boxed{y = ce^{-\cos x}, \quad c \neq 0 \text{ is constant}}$$

Note $y \equiv 0$ also satisfies ①, So $\boxed{y \equiv 0}$ is also a solution.

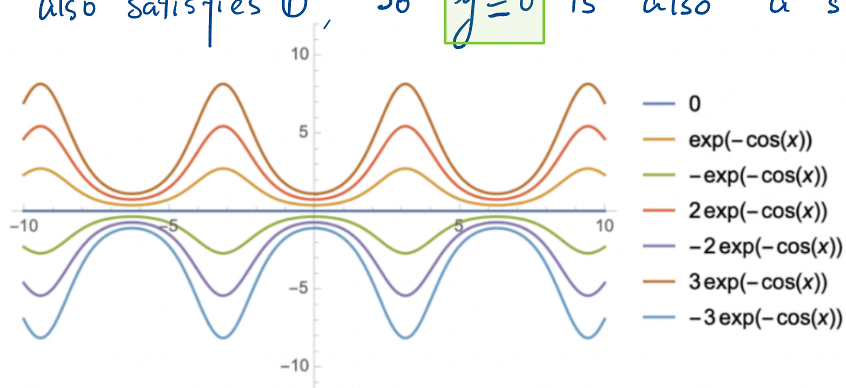

Figure. The solution curves for $\dfrac{dy}{dx} = y\sin x$.

General Separable Equations

In general, the first-order differential equation $\frac{dy}{dx} = f(x, y)$ is separable if $f(x, y)$ can be written as the product of a function of x and a function of y:

$$\frac{dy}{dx} = f(x, y) = g(x)k(y) \tag{2}$$

- If $k(y) \neq 0$, then we can write

$$\frac{dy}{k(y)} = g(x)dx \tag{3}$$

- To solve the differential equation we simply integrate both sides:

$$\int \frac{dy}{k(y)} = \int g(x)dx + C$$

- Note we also need to check if $k(y) = 0$ gives us a solution.

Implicit, General, and Singular Solutions

- **General solution:** A solution of a differential equation that contains an "arbitrary constant" C.

  For example, in **Example 1**, $y = Ce^{-\cos x}, C \neq 0$ is a constant is a general solution.

- **Singular solution:** Exceptional solutions cannot be obtained from the general solution.

  In **Example 1**, $y = 0$ is a singular solution.

- **Implicit solution** The equation $K(x, y) = 0$ is commonly called an implicit solution of a differential equation if it is satisfied (on some interval) by some solution $y = y(x)$ of the differential equation.

  For example, in **Example 1**, $\ln|y| = e^{-\cos x} + C$ is an implicit solution

Example 2. Find solutions of the differential equation $2\sqrt{x}\frac{dy}{dx} = \sqrt{1-y^2}.$

ANS: Note $1-y^2 \geq 0 \Rightarrow -1 \leq y \leq 1$

If $\sqrt{1-y^2} \neq 0$, $x \neq 0$, we have

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$$

$$\Rightarrow \quad \sin^{-1} y = \sqrt{x} + C$$

$$\Rightarrow \quad y(x) = \sin(\sqrt{x} + C)$$

If $\sqrt{1-y^2} = 0$, $y(x) \equiv \pm 1$, which also satisfy the given equation

So the equation has general solution

$$\boxed{y(x) = \sin(\sqrt{x} + C)}$$

and singular solutions

$$\boxed{y(x) \equiv \pm 1}$$

Example 3. Find the particular solution if the initial value problem

separable

$$2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}, \qquad y(5) = 2.$$

ANS: We have

$$\int 2y\,dy = \int \frac{x}{\sqrt{x^2-16}}\,dx$$

$$\int \frac{x}{\sqrt{x^2-16}}\,dx$$

Let $u = x^2-16$, then $du = 2x\,dx$

$\Rightarrow x\,dx = \frac{1}{2}du$

Thus $\int \frac{x}{\sqrt{x^2-16}}\,dx = \int \frac{\frac{1}{2}du}{\sqrt{u}} = \sqrt{u} + C$

$= \sqrt{x^2-16} + C$

$$\Rightarrow y^2 = \sqrt{x^2-16} + C$$

As $y(5) = 2$,

$$4 = 2^2 = \sqrt{5^2-16} + C = 3 + C$$

$$\Rightarrow C = 1.$$

So

$$y^2 = \sqrt{x^2-16} + 1 \quad \text{(implicit solution)}$$

or

$$\boxed{y = \pm\sqrt{\sqrt{x^2-16} + 1}}$$

Natural Growth and Decay

The differential equation

$$\frac{dx}{dt} = kx \quad (k \text{ a constant }) \tag{4}$$

serves as a mathematical model for a remarkably wide range of natural phenomena.

Population Growth

- Suppose that $P(t)$ is the size of a population, say of humans, or insects, or bacteria, having constant birth and death rates β and δ.
- These rates are measured in births or deaths per individual per unit of time.
- Then during a short time interval Δt, there occur roughly

$$\beta P(t)\Delta t \quad \text{births} \tag{5}$$

  and

$$\delta P(t)\Delta t \quad \text{deaths.} \tag{6}$$

- So the change in $P(t)$ is approximately

$$\Delta P \approx (\beta - \delta)P(t)\Delta t \tag{7}$$

  and therefore

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = kP \tag{8}$$

  where $k = \beta - \delta$.

- Thus the population $P(t)$ satisfies our differential equation

$$\frac{dP}{dt} = kP \tag{9}$$

Example 4 (Population growth) In a certain culture of bacteria, the number of bacteria increased sixfold in 10h. How long did it take for the population to double?

ANS: Let $x(t)$ be the population at time t

$$\frac{dx(t)}{dt} = kx, \qquad x(10) = 6x(0) = 6x_0$$

$$\Rightarrow \int \frac{dx}{x} = \int k\,dt$$

$$\Rightarrow \ln x = kt + C_1 \quad (x>0)$$

$$\Rightarrow x = e^{\ln x} = e^{kt+C_1} = \underbrace{e^{C_1}}_{=C} e^{kt} = C\cdot e^{kt}$$

$$\Rightarrow x(t) = Ce^{kt}$$

As $x(0) = x_0$, $x(0) = Ce^{k\cdot 0} = x_0 \Rightarrow C = x_0$

As $x(10) = 6x_0$, $x(10) = \cancel{x_0} e^{10k} = 6\cancel{x_0}$

$$\Rightarrow e^{10k} = 6 \Rightarrow k = \frac{\ln 6}{10}. \quad \text{So } x(t) = x_0 e^{\frac{\ln 6}{10}t}$$

The population will double when

$$x(t) = \cancel{x_0} e^{\frac{\ln 6}{10}t} = 2\cancel{x_0}$$

$$\Rightarrow e^{\frac{\ln 6}{10}t} = 2 \Rightarrow \ln e^{\frac{\ln 6}{10}t} = \frac{\ln 6}{10}t = \ln 2$$

$$\Rightarrow t = \frac{10\ln 2}{\ln 6}\,h \approx \boxed{3.87\,h}$$

Radioactive Decay

- Consider a sample of material that contains $N(t)$ atoms of a certain radioactive isotope at time t.
- It has been observed that a constant fraction of these radioactive atoms will spontaneously decay during each unit of time.
- Thus mathematically, the sample behaves like a population with a constant death rate and no births, leading once again
  to our differential equation

$$\frac{dN}{dt} = -kN \tag{10}$$

- The value of k depends on the particular radioactive isotope.

Example 5 (Natural decay) A specimen of charcoal found at Stonehenge turns out to contain 63% as much ^{14}C as a sample of present-day charcoal of equal mass. What is the age of the sample?

Note for $^{14}\mathrm{C}, \quad k \approx 0.0001216$

ANS: $\frac{dN}{dt} = -kN, \quad N(0) = N_0$

We need to find t when $N(t) = 0.63 N_0$

$$\int \frac{dN(t)}{N} = -\int k\,dt$$

$$\Rightarrow \ln N = -kt + C_1$$

$$\Rightarrow N(t) = Ce^{-kt}$$

As $N(0) = N_0$, $N(0) = Ce^{-k\cdot 0} = C = N_0$

We solve $N(t) = \cancel{N_0}e^{-kt} = 0.63\cancel{N_0}$ for t

$$\Rightarrow e^{-kt} = 0.63 \Rightarrow -kt = \ln 0.63$$

$$\Rightarrow t = -\frac{\ln 0.63}{0.0001216} \approx \boxed{3800 \text{ years}}$$

Cooling and Heating

According to ***Newton's law of cooling***, the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature A is proportional to the difference $A - T$, i.e.,

$$\frac{dT}{dt} = k(A - T) \tag{11}$$

where k is a positive constant.

Example 6

- A 4-lb roast, initially at $50°\mathrm{F}$, is placed in a $375°\mathrm{F}$ oven at $5:00$ P.M.
- After 75 minutes it is found that the temperature $T(t)$ of the roast is $125°\mathrm{F}$.
- When will the roast be $150°\mathrm{F}$, that is, medium rare?

ANS: · We take t in minutes, with $t=0$ corresponding to 5P.M.

· We also assume that any instant temperature $T(t)$ of the roast is uniform throughout.

· We have

$T(0) = 50°F$, $T(75) = 125°F$. $T(t) < 375°F$

$$\frac{dT}{dt} = k(375 - T) \quad (\text{sep.})$$

$$\Rightarrow \int \frac{dT}{375-T} = \int k\,dt$$

$$\Rightarrow -\int \frac{d(375-T)}{375-T} = \int k\,dt$$

$$\Rightarrow -\ln(375-T) = kt + C_1$$

$$\Rightarrow \ln(375-T) = -kt - C_1$$

$$\Rightarrow 375 - T = \overset{C}{\cancel{e^{-C_1}}}\, e^{-kt} = Ce^{-kt}$$

$$\Rightarrow T(t) = 375 - Ce^{-kt}$$

Since $T(0) = 50°F$

$$T(0) = 50 = 375 - C$$

$$\Rightarrow \quad C = 325$$

$$T(t) = 375 - 325\, e^{-kt}$$

Since $T(75) = 125$

$$\Rightarrow \quad 125 = 375 - 325\, e^{-75k}$$

$$\Rightarrow \quad 325\, e^{-75k} = 375 - 125 = 250$$

$$\Rightarrow \quad e^{-75k} = \frac{250}{325}$$

$$\Rightarrow \quad -75k = \ln \frac{250}{325}$$

$$\Rightarrow \quad k = -\frac{1}{75} \ln \frac{250}{325} \approx 0.0035$$

The question asks us to find t when $T(t) = 150$

Set

$$375 - 325\, e^{-0.0035t} = 150$$

$$\Rightarrow \quad t = -\frac{1}{0.0035} \ln \frac{225}{325} \approx 105 \text{ min}$$

So the roast should be removed at about

$\boxed{\text{6:45 PM}}$