1.3 Slope Fields and Solution Curves

More general differential equations

- Now we consider differential equations of the form

$$\frac{dy}{dx} = f(x, y)$$

where the right-hand function $f(x, y)$ involves both the independent variable x and the dependent variable y.

- We might think of integrating both sides, say

unknown.

$$y(x) = \int f(x, y(x))dx + C$$

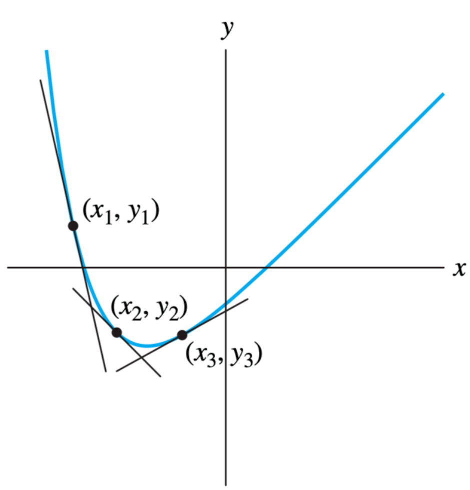
However, because we do not know the function $y(x)$, we cannot evaluate this integral.

- Using *graphical* and *numerical* methods we can construct approximate solutions of differential equations.

Slope Fields and Graphical Solutions

Differential Equations and Slopes

- We can view solutions of the differential equation $y' = f(x, y)$ in a simple geometric way.
- It rests on the general fact that *first derivatives give slopes of tangent lines*.
- Thus at each point (x, y) of the xy -plane, the value of $f(x, y)$ determines a slope $m = f(x, y)$
- A **solution curve** of the differential equation is then simply a curve in the xy -plane whose tangent line at (x, y) has slope $f(x, y)$



Consider the diff. eqn.

$$y' = x - y$$

$\Longrightarrow$ This is a solution curve for the eqn $y' = x - y$ together with the tangent lines have

- slope $m_1 = x_1 - y_1$ at the pt (x_1, y_1)
- slope $m_2 = x_2 - y_2$ at the pt (x_2, y_2)
- slope $m_3 = x_3 - y_3$ at the pt (x_3, y_3)

Graphical Method

- This leads to a graphical method for constructing approximate solutions of the differential equation

$$\frac{dy}{dx} = f(x,y)$$

- First we choose a representative collection of points (x, y) in the plane.
- Through each point (x, y) we draw a short line segment having slope $m = f(x, y)$
- All these line segments taken together constitute a **slope field** for the equation $y' = f(x,y)$

$x \backslash y$	-4	-3	-2	-1	0	1	2	3	4
-4	0	-1	-2	-3	-4	-5	-6	-7	-8
-3	1	0	-1	-2	-3	-4	-5	-6	-7
-2	2	1	0	-1	-2	-3	-4	-5	-6
-1	3	2	1	0	-1	-2	-3	-4	-5
0	4	3	2	1	0	-1	-2	-3	-4
1	5	4	3	2	1	0	-1	-2	-3
2	6	5	4	3	2	1	0	-1	-2
3	7	6	5	4	3	2	1	0	-1
4	8	7	6	5	4	3	2	1	0

Figure Values of the slope $y' = x - y$ for $-4 \leq x, y \leq 4$

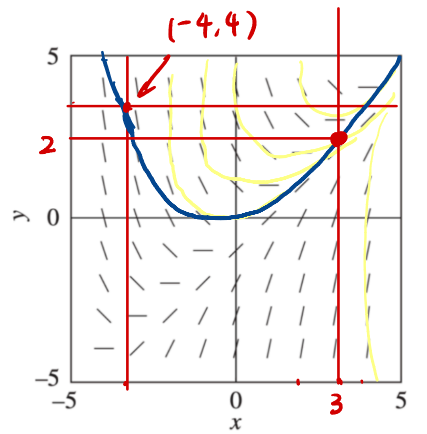


Figure Slope field for $y' = x - y$ for the table of slopes

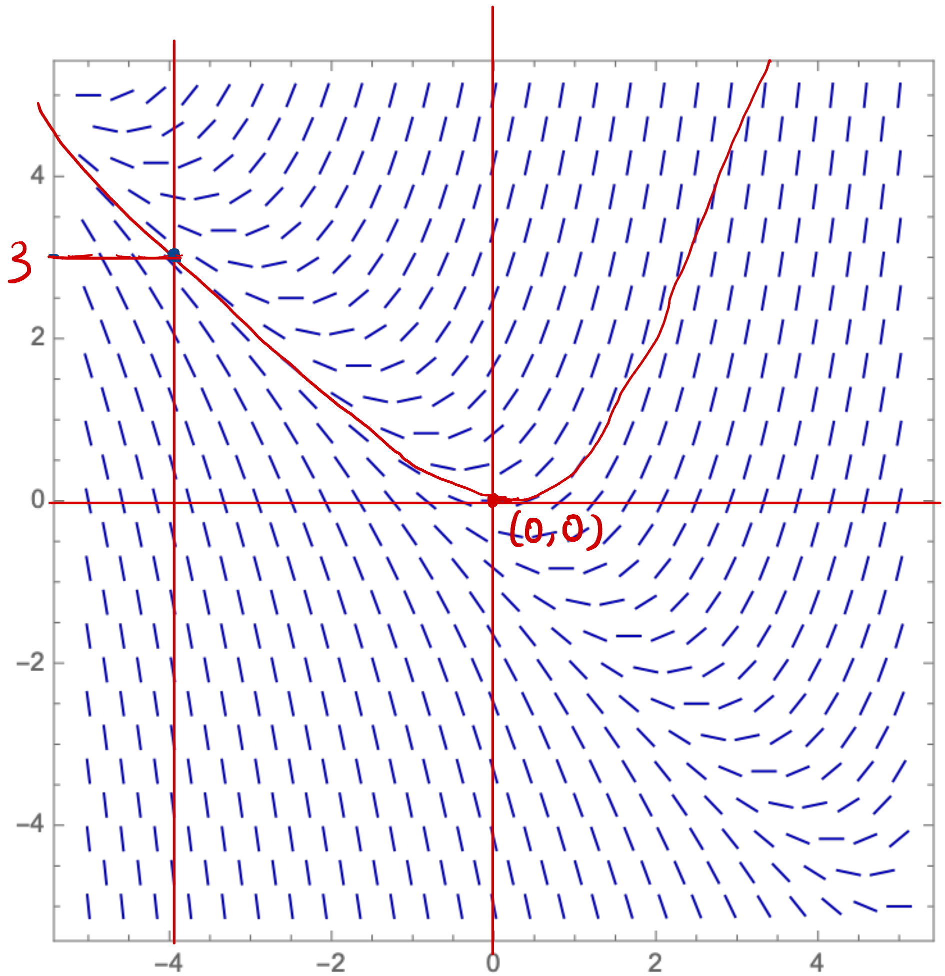
Question: Let $y(x)$ be the solution of the initial value problem $y' = x - y, \underline{y(-4) = 4}.$

(-4, 4)

Can you estimate the value of $y(3)$? ≈ 2 .

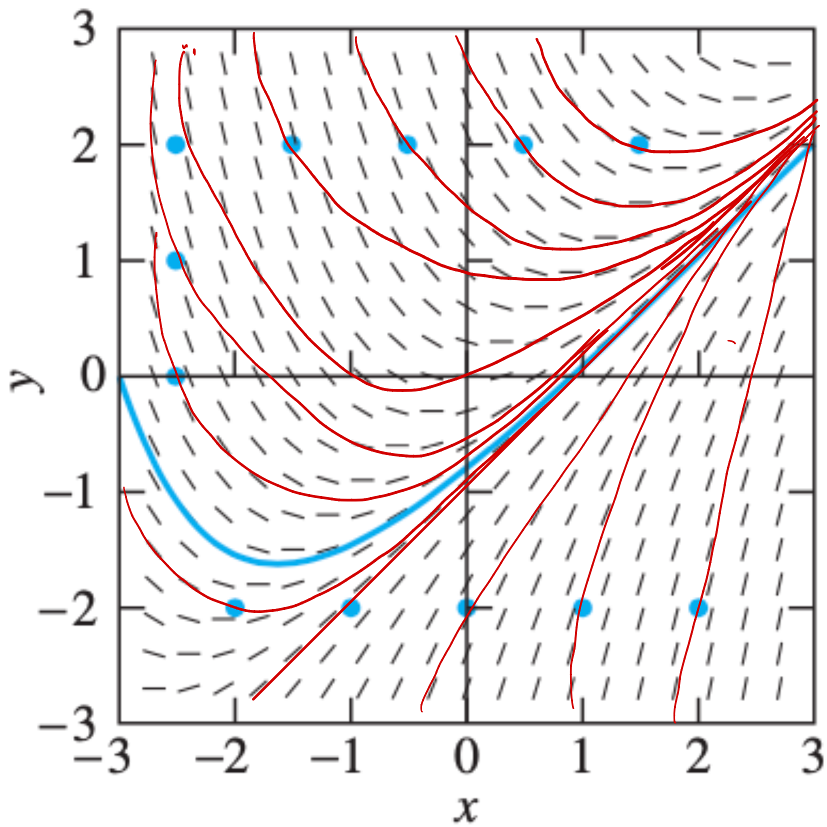
Example 1 In the following problem, first construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

$y' = x + y, \quad y(0) = 0; \quad y(-4) = ?$ ≈ 3



Example 2 We have provided the slope field of the indicated differential equation, together with one or more solution curves. Sketch likely solution curves through the additional points marked in each slope field.

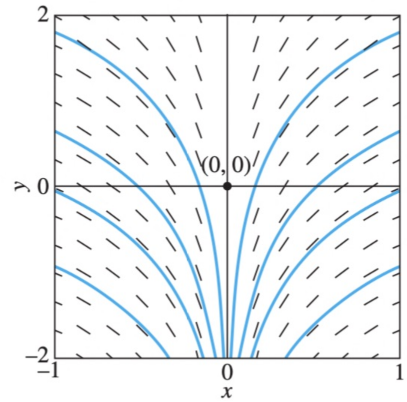
$$\frac{dy}{dx} = x - y$$



Existence and Uniqueness of Solutions

Example 3 Does the following initial value problem have a solution? No

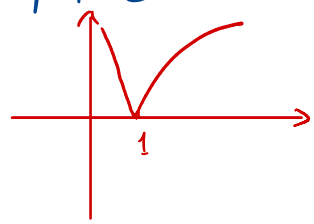
$$y' = \frac{1}{x},\ y(0) = 0$$ Failure of existence



slope field and typical solution curves for $y' = \frac{1}{x}$

ANS: $\frac{dy}{dx} = \frac{1}{x}$

$\Rightarrow \quad y = \int \frac{1}{x}\,dx = \ln|x| + C$

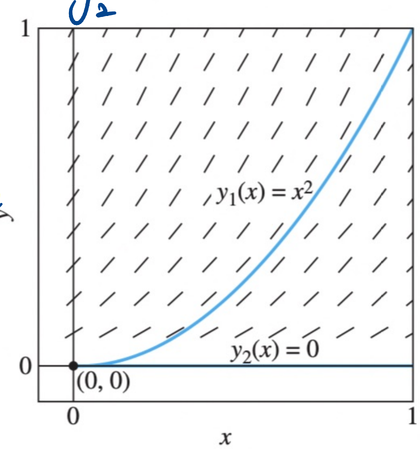


Example 4 Verify $y_1 = x^2$ ($x > 0$) and $y_2 \equiv 0$ satisfy the following initial value problem.

$$y' = 2\sqrt{y},\ y(0) = 0$$ Failure of uniques.

ANS: $y_1' = 2x \stackrel{?}{=} 2\sqrt{y_1} = 2\sqrt{x^2} = 2x$, Also $y_1(0) = 0^2 = 0$

$y_2' = 0 = 2\sqrt{y_2} = 2\sqrt{0} = 0$ Also $y_2(0) = 0$



Slope field and two different solution curves for the initial value problem $y' = 2\sqrt{y},\ y(0) = 0$

Question: When we have unique solution for the initial value problem?

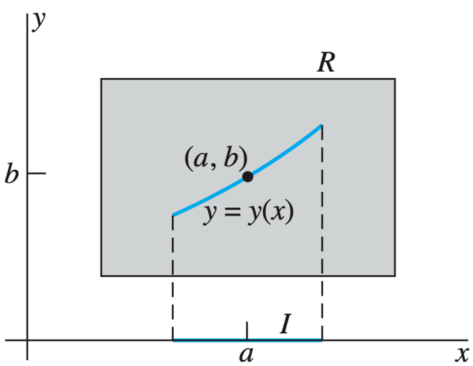
THEOREM 1 Existence and Uniqueness of Solutions

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the xy -plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has one and only one solution that is defined on the interval I. (As illustrated in Fig. 1 the solution interval I may not be as "wide" as the original rectangle R of continuity. See the **Remark** below)

Remark: If the hypotheses of Theorem 1 are not satisfied, then the initial value problem $y' = f(x, y)$, $y(a) = b$ may have either no solutions, finitely many solutions, or infinitely many solutions.



Example 5 (SPRING 2018 final. Question 3)

What is the *largest open interval* in which the solution of the initial value problem

$$\begin{cases} t^2 y' + \dfrac{\ln|t-1|}{e^{t-2}} y = \dfrac{t-5}{\sin(t-4)} \\ y(3) = \pi \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

A. (0,4)

B. (2,5)

C. $(4-\pi, 5)$

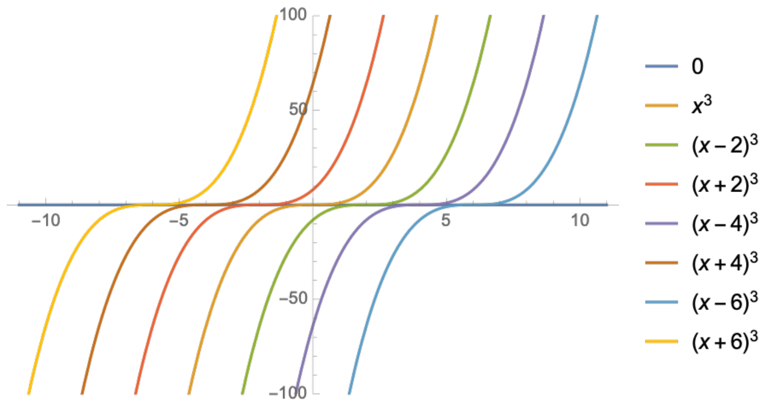
D. (1,4)

E. $(4-\pi, 4)$

Example 6 Verify that if c is a constant, then the function defined piecewise by

$$y = \begin{cases} 0 & \text{for } x \leq c, \\ (x-c)^3, & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 3y^{2/3}$ for all x. Can you also use the "left half" of the cubic $y = (x-c)^3$ in piecing together a solution curve of the differential equation? Sketch a variety of such solution curves. Is there a point (a, b) of the xy-plane such that the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ has either no solution or a unique solution that is defined for all x? Reconcile your answer with Theorem 1.



Example 7 Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} -1 & \text{if } x \leq c - \pi/2 \\ \sin(x - c) & \text{if } c - \pi/2 < x < c + \pi/2 \\ 1 & \text{if } x \geq c + \pi/2 \end{cases}$$

satisfies the differential equaiton $y' = \sqrt{1 - y^2}$ for all x. Sketch a variety of such solution curves for different values of c. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = \sqrt{1 - y^2}$, $y(a) = b$ has.