1.2 Integrals as General and Particular Solutions

Integrating Both Sides

- The first-order equation $\frac{dy}{dx} = f(x, y)$ takes an especially simple form if the right-hand-side function f does not actually involve the dependent variable y, so

$$y' = \frac{dy}{dx} = f(x) \tag{1}$$

Note $f(x)$ does not involve y

$$\Rightarrow dy = f(x)dx \Rightarrow \int dy = y = \int f(x)dx + c$$

- In this special case we need only integrate both sides of the equation to obtain

$$\boxed{y(x) = \int f(x)dx + C} \tag{2}$$

- This is a general solution of the differential equation, meaning that it involves an arbitrary constant C, and for every choice of C it is a solution of the differential equation.

Example 1 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

$$\frac{dy}{dx} = (x-2)^2;\ y(2) = 1$$

ANS: We have $\frac{dy}{dx} = x^2 - 4x + 4$

Integrate both sides, we have

$$y = \int (x^2 - 4x + 4)\,dx + C$$

$$= \frac{1}{3}x^3 - \frac{4}{2}x^2 + 4x + c$$

$$\Rightarrow \boxed{y = \frac{1}{3}x^3 - 2x^2 + 4x + c}$$

general solution

Since $y(2) = 1$.

$$y(2) = \frac{1}{3}2^3 - 2\cdot 2^2 + 4\cdot 2 + C = 1$$

$$\Rightarrow \frac{8}{3} - 8 + 8 + C = 1$$

$$\Rightarrow C = -\frac{5}{3}$$

So

$$\boxed{y = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{5}{3}}$$

Recall

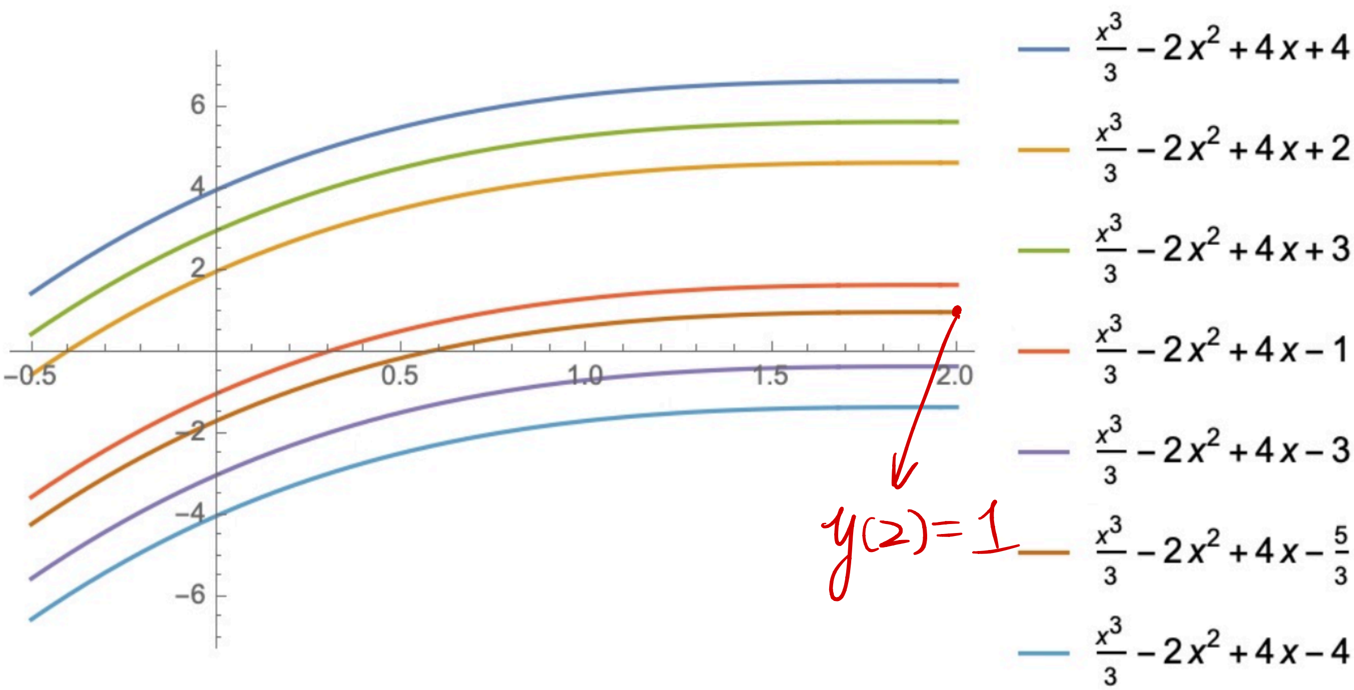
$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

Let's look at the graph of the general solution

$y = \frac{1}{3}x^3 - 2x^2 + 4x + C$

and the particular solution

$y = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{5}{3}$



Example 2 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

$$\frac{dy}{dx} = \frac{1}{\sqrt{x+4}}; \; y(5) = 7$$

ANS: Integrate both sides,

$$y = \int \frac{1}{\sqrt{x+4}} dx$$

$$= \int (x+4)^{-\frac{1}{2}} dx$$

$$= \frac{1}{1-\frac{1}{2}} (x+4)^{-\frac{1}{2}+1} + C$$

$$= 2 \cdot (x+4)^{\frac{1}{2}} + C = 2\sqrt{x+4} + C$$

As $y(5) = 7$,

$$y(5) = 2\overset{3}{\cancel{\sqrt{5+4}}} + C = 7$$

$$\Rightarrow C = 1$$

Example 3 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}; \; y(0) = 0 \tag{3}$$

ANS: We have

$$y = \int \frac{1}{\sqrt{1-x^2}} dx$$

$$= \sin^{-1} x + C$$

As $y(0) = 0$, $C = 0$

Thus $y = \sin^{-1} x$

Exercise 4 *Solution will be posted in the complete notes.*

Find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

$$a(t) = 2t + 1,\ v_0 = -7,\ x_0 = 4.$$

ANS: Note $a(t) = \frac{dv}{dt} = 2t+1 \Rightarrow v(t) = \int (2t+1)\,dt + C_1 = t^2 + t + C$

As $v(0) = -7$, $v(0) = C_1 = -7$.

We have

$$v(t) = t^2 + t - 7$$

As $\frac{dx(t)}{dt} = v(t) = t^2 + t - 7$

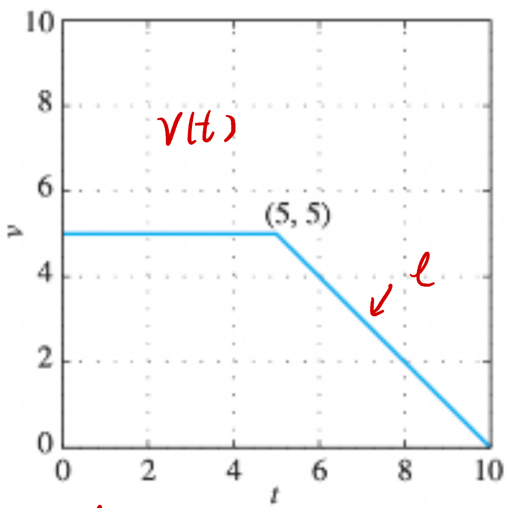
$$\Rightarrow \quad x(t) = \int (t^2 + t - 7)\,dt + C_2 = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 7t + C_2$$

As $x(0) = 4$, $C_2 = 4$.

Thus $\boxed{x(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 7t + 4}$

$x(0) = 0$

Example 5 A particle starts at the origin and travels along the x-axis with the velocity function $v(t)$ whose graph is shown in the Figure below. Sketch the graph of the resulting position function $x(t)$ for $0 \le t \le 10$.



ANS: From the graph, we know.

$$v(t) = \begin{cases} 5, & 0 \le t \le 5 \\ 10 - t, & 5 \le t \le 10 \end{cases}$$

Note l passes two pts $(5,5), (10,0)$

Thus l satisfies

$$v = \frac{5-0}{5-10}(t - 10)$$

$$v(t) = 10 - t,\ 5 \le t \le 10$$

Then $\frac{dx(t)}{dt} = v(t)$

$$\Rightarrow \quad x(t) = \int v(t)\,dt = \begin{cases} 5t + C_1, & 0 \le t \le 5 \\ 10t - \frac{1}{2}t^2 + C_2, & 5 \le t \le 10 \end{cases}$$

Since $x(0) = 0$, $C_1 = 0$

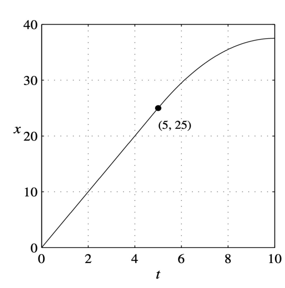
The continuity of $x(t)$ requires $x(t) = 5t$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ agree when $t = 5$.

That is.

$$5 \cdot 5 = 10 \cdot 5 - \frac{1}{2} \cdot 5^2 + C_2$$

$$\Rightarrow C_2 = 25 - 50 + \frac{25}{2} = -\frac{25}{2}$$

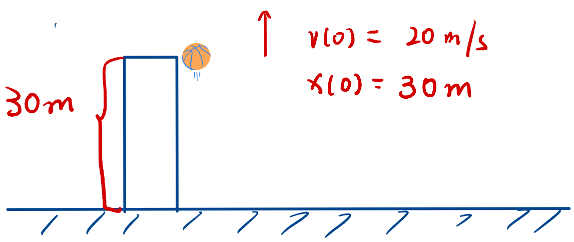
$$x(t) = \begin{cases} 5t, & 0 \le t \le 5 \\ 10t - \frac{1}{2}t^2 - \frac{25}{2}, & 5 \le t \le 10 \end{cases}$$



Example 6 A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.

(a) Find the maximum height above the ground that the ball reaches. $\Leftrightarrow$ What is $x(t)$ when $v(t) = 0$

(b) Assuming that the ball misses the building on the way down, find the time that it hits the ground. $\Leftrightarrow$ what is t when $x(t)=0$



ANS: Let $x(t)$ be the position of the ball above the ground.

Then $x(0) = 30$ m.

$$\frac{dv(t)}{dt} = -g \quad (g \approx 9.81)$$

$$\Rightarrow v(t) = \int -g\,dt = -gt + C_1$$

As $v(0) = 20\,m/s$, $C_1 = 20$

Thus $v(t) = -gt + 20$

Let $v(t) = -gt + 20 = 0 \Rightarrow t = \frac{20}{g} \approx 2.0387s.$

What is $x(2.0387) = ?$

$$\frac{dx(t)}{dt} = v(t) = -gt + 20$$

$$\Rightarrow x(t) = \int (-gt + 20)\,dt$$

$$\Rightarrow x(t) = -\frac{1}{2}gt^2 + 20t + C_2$$

As $x(0) = 30$, $C_2 = 30$

Thus $x(t) = -\frac{1}{2}gt^2 + 20t + 30$

$x(2.0387) \approx 50.4\,m$

(b) Let $x(t) = -\frac{1}{2}gt^2 + 20t + 30 = 0$

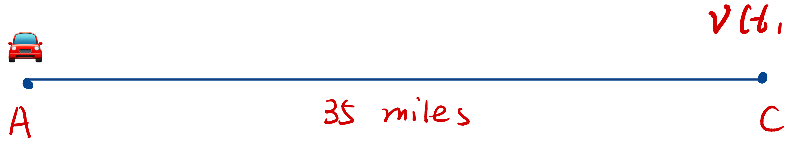
$\Rightarrow t = -1.16630$ or $\boxed{t = 5.24382s}$

$V(0)=0$ assume it is a

Example 7 At noon a car starts from rest at point **A** and proceeds with constant acceleration along a straight road toward point **C** , 35 miles away. If the constantly accelerated car arrives at **C** with a velocity of 60 mi/h, at what time does it arrive at **C** ?

Assume at time t_1 $\quad x(t_1)=35$ mi

$v(t_1)=60$ mi/h



ANS: Let $x(t)$ be the distance of the car from point A

Then $x(0)=0$.

Let $v(t)$ be the velocity of the car at t.

$$\frac{dv}{dt}=a \Rightarrow v=\int a\,dt+C_1 = at+C_1$$

As $v(0)=0$, $C_1=0$. Then $v(t)=at$

As $\frac{dx(t)}{dt}=v(t)=at$

$$\Rightarrow x(t)=\int at\,dt = \frac{1}{2}at^2+C_2$$

As $x(0)=0$, $C_2=0$

Thus $x(t)=\frac{1}{2}at^2$

We know $\quad x(t_1)=\frac{1}{2}at_1^2=35$

$v(t_1)=at_1=60$

Then $$\frac{x(t_1)}{v(t_1)}=\frac{\frac{1}{2}\cancel{a}t_1^2}{\cancel{a}t_1}=\frac{1}{2}t_1=\frac{35}{60} \Rightarrow t_1=\frac{70}{60}=1\text{ h }10\text{ min}$$

$\boxed{1:10\text{pm}}$