1.1 Differential Equations and Mathematical Models

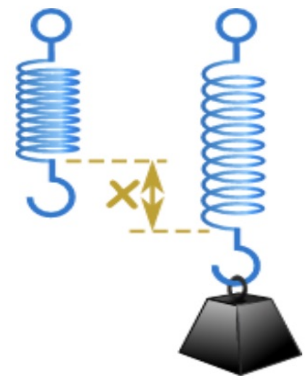
Differential Equations

Changing Quantities

- The laws of the universe are written in the language of mathematics.
- Algebra is sufficient to solve many static problems,
- but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Number of rabbits

Saving account balance



Position of an object

$$\text{Rate of Change} = \frac{\text{change of } x}{\text{change of } t} = \frac{\Delta x}{\Delta t} \qquad \frac{dx}{dt} = x'$$

Derivative as Rate of Change

- Because the derivative $dx/dt = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t.
- It is natural that equations involving derivatives are frequently used to describe the changing universe.
- What is a **differential equation**?

  An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

Order of a diff. eqn. is the order of the highest derivative present in the equation

Examples of differential equations

- The equation

$$\frac{dx}{dt} = x^2 + t^2 \quad \text{(first order diff. eqn)} \tag{1}$$

involves the unknown function $x(t)$ and its first derivative $x'(t)$.

- The equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0 \quad \text{(second order diff. eqn)} \tag{2}$$

involves the unknown function $y(x)$ and its first two derivatives.

Goals of the Study of Differential Equations

Three Goals

The study of differential equations has three principal goals:

1. To **discover** the differential equation that describes a specified physical situation.
2. To **find** - either exactly or approximately - the appropriate solution of that equation.
3. To **interpret** the solution that is found.

Unknowns

- In algebra, we typically seek the unknown numbers that satisfy an equation such as

$$x^3 + 7x^2 - 11x + 41 = 0.$$

- By contrast, in solving a differential equation, we are challenged to find the unknown functions $y = y(x)$ for which an identity such as $y'(x) = 2xy(x)$ - that is, the differential equation

$$\frac{dy}{dx} = 2xy \tag{3}$$

holds on some interval of real numbers.

- Ordinarily, we will want to find *all solutions* of the differential equation, if possible.

Overview: Summary from the "Useful links"

Example 1 Substitute $y = e^{rt}$ into the given differential equation to determine all values of the constant r for which $y = e^{rt}$ is a solution of the equation.

$$y'' + 3y' - 4y = 0$$

Chain Rule

$$[f(g(t))]' = f'(g(t)) \cdot g'(t)$$

Ex: $[\sin(2t)]' = (\cos 2t)(2t)'$

$= 2\cos 2t$

ANS: If $y = e^{rt}$, then

$$y' = (e^{rt})' = r \cdot e^{rt}$$

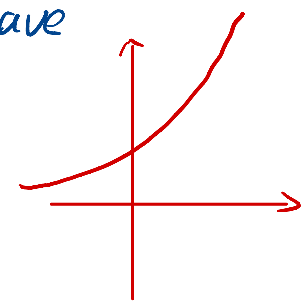
$$y'' = (re^{rt})' = r(e^{rt})' = r^2 e^{rt}$$

Substitute y, y', y'' to the given eqn, we have

$$r^2 e^{rt} + 3re^{rt} - 4e^{rt} = 0$$

$$\Rightarrow \overset{\neq 0}{e^{rt}}(r^2 + 3r - 4) = 0$$

Note e^{rt} can't be 0.



$$\Rightarrow r^2 + 3r - 4 = 0$$

$$\Rightarrow (r+4)(r-1) = 0 \Rightarrow \boxed{r = -4} \text{ or } \boxed{r = 1}$$

Example 2 Verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial conditon.

$y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $\underline{y(0) = 10}$

ANS: $LHS = y' = (Ce^{-x} + x - 1)' = -Ce^{-x} + 1$

$\parallel$

$RHS = x - y = x - Ce^{-x} - x + 1 = -Ce^{-x} + 1$

Thus y satisfies the given diff. eqn.

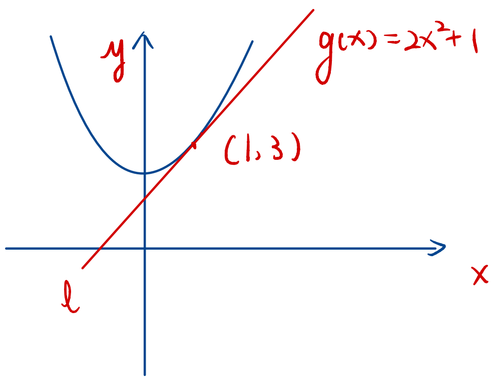
Since $y(0) = 10$

$$y(0) = Ce^{-0} + 0 - 1 = C - 1 = 10$$

$$\Rightarrow \boxed{C = 11}$$

Geometric properties of functions

Review: Let $g(x) = 2x^2 + 1$ and let ℓ be the line tangent to the graph of $g(x)$ at point $(1, 3)$. What is the slope of ℓ?

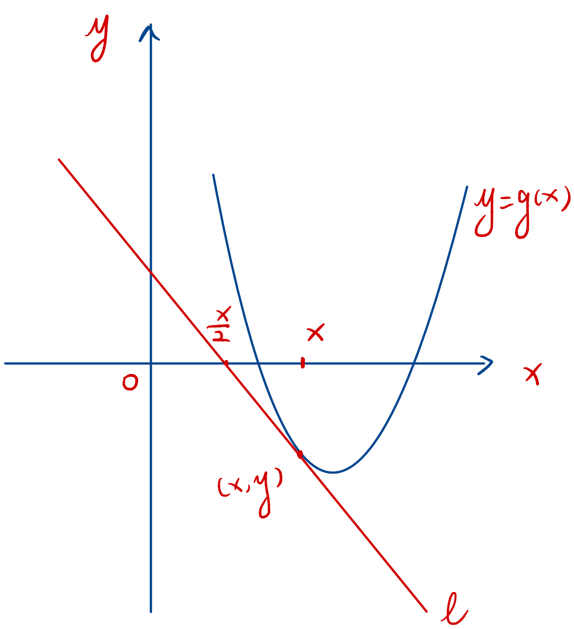


ANS: As $g'(x) = 4x$

the slope of ℓ is $g'(1) = 4$

Example 3 A function $y = g(x)$ is described by the following geometric property of its graph. Write a differential equation of the form $\frac{dy}{dx}$ having the function g as its solution (or as one of its solutions).

The line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $(\frac{x}{2}, 0)$.



ANS: What is the slope m of line ℓ?

· On one hand, the point (x, y) and $(\frac{x}{2}, 0)$ are on the line ℓ.

$$m = \frac{y - 0}{x - \frac{x}{2}} = \frac{y}{\frac{x}{2}} = \frac{2y}{x}$$

· On the other hand, the slope of the tangent line ℓ to $g(x)$ at point (x, y)

$$m = g'(x)$$

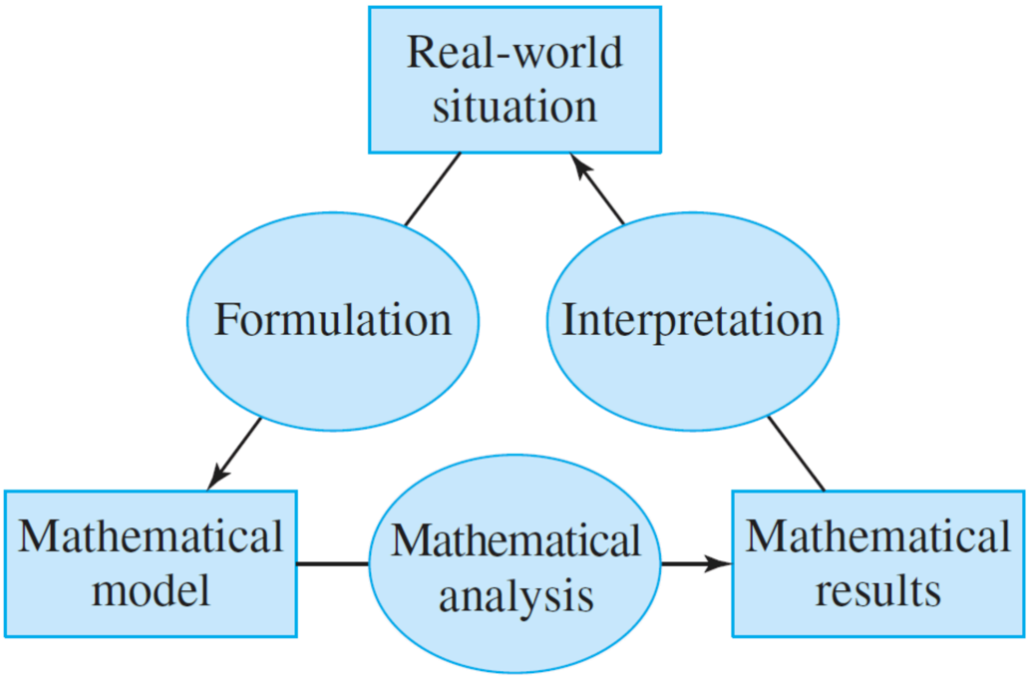
Thus, we have

$$\boxed{g'(x) = \frac{2y}{x}}$$

or $\frac{dy}{dx} = \frac{2y}{x}$

Mathematical Models

The Process of Mathematical Modeling



- The following example (**Example 4**) illustrates the process of translating scientific laws and principles into differential equations.
- We will see more mathematical models throughout this semester.

N'

Example 4 In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not. Set up a differential equation for N.

multiply with a constant k function of t.

ANS: We know:

- The number of persons with disease: $N(t)$
- Rate of change of $N(t)$: $\frac{dN(t)}{dt} = N'(t)$
- The number who do not have the disease: $P - N(t)$

$$\frac{dN(t)}{dt} = k \cdot N(t) \cdot (P - N(t))$$

$$\Rightarrow \quad \frac{dN}{dt} = k \cdot N \cdot (P - N)$$